Current topics in Few-Body Problems

Beyond the horizon of the three-body Faddeev equations

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JAEA_Tutorial*15, @ JAEA_Nov 20 2015

This talk is about the few-body problems by the three-body Faddeev equations in 50-years.

1) Applied to hadron physics, Nuclear physics, atomic physics,
2) from three-body to A-body problems
3) the nuclear potentials check
4) Three-body force estimation
5) Relativistic extension
6) Applied for fundamental problems
7) Calculation methods & analysis of Exp.data.
8) Long range Coulomb problem has been investigated.

INTRODUCTION

Benefits of the three-body Faddeev equations:

1) Describe:

Full Born series by the integral equation

> perturbation, etc.

Full kinematics > pick-up, knock-on,

stripping, heavy particle stripping

(*with the large momentum transfer*) , etc.

Therefore, the Faddeev approach is a opposite end of the recoilless interaction in the many-body system

(*where the particle's creation and annihilation or particle and hole creation operators are used*.)

Contain full interactions
(*two-body amplitude, 3BF amplitude*)
> two-body potential, 3BF-potential

2) Extension (generalization) :
Three-body → A-body
(*automatic*,
***but the increase of numerical burden and the progress of hardware are always put in the balance.*)**

3) Reduction:
Three-body → Two-body
(*the multi-channel Lippmann-Schwinger equations below the break up threshold*)

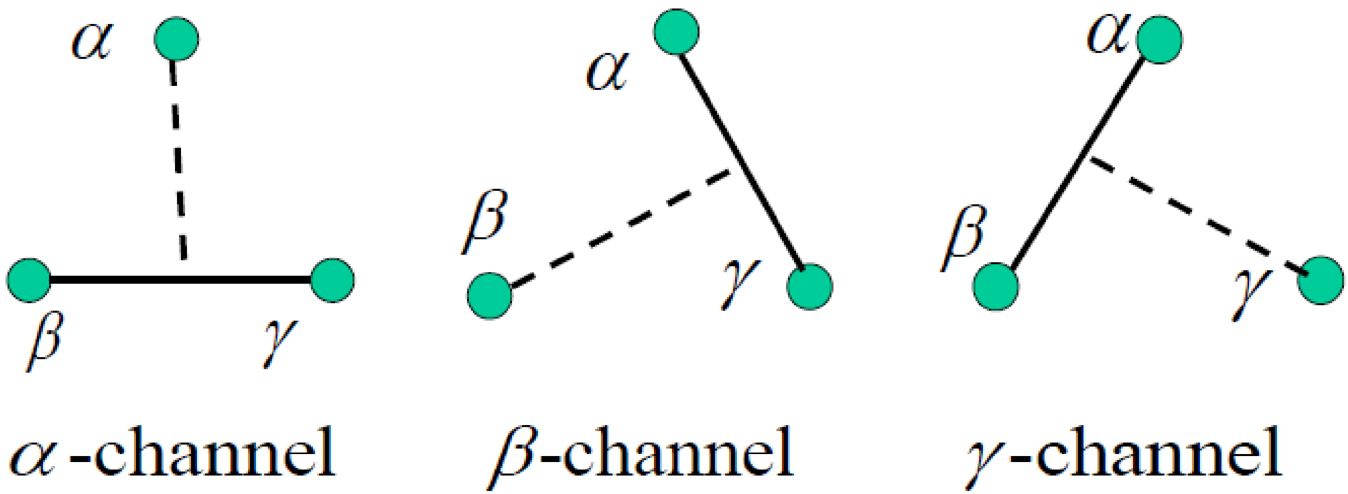
The cluster formation depends on the threshold energy:
(*the multi-channel few-body Faddeev equations with the few-cluster force are constructed*)

4) Recent development :

a) Research of the threshold behavior by the Faddeev's approach makes an offer a new frontier.
b) The Coulomb interaction is now treated in the Faddeev equations.

1. Three-body Faddeev equation

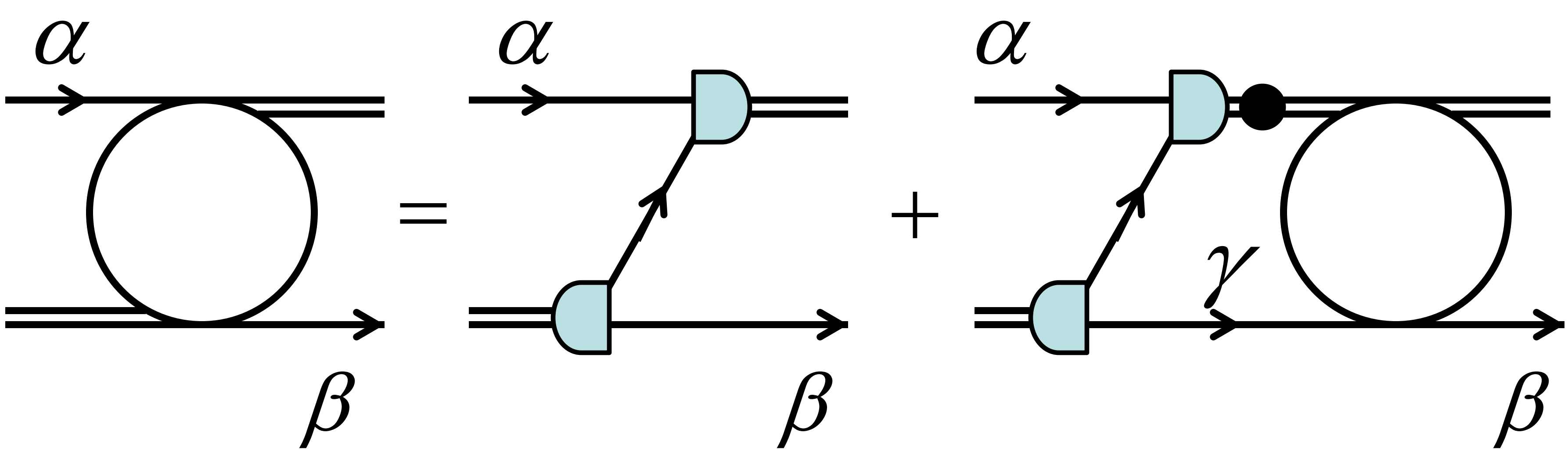
1. L.D. Faddeev,
Soviet Phys.-JETP 12 (1961) 1014; Soviet Phys. Dokl. 6 (1961)384; ibid. 7 {1963) 600.
2. L. D. Faddeev,
Mathematical aspects of the three-body problem in the quantum scattering theory. (Israel Program for Scientific Translation,
Jerusalem, 1965,
distributed by Oldbourne Press, London.)

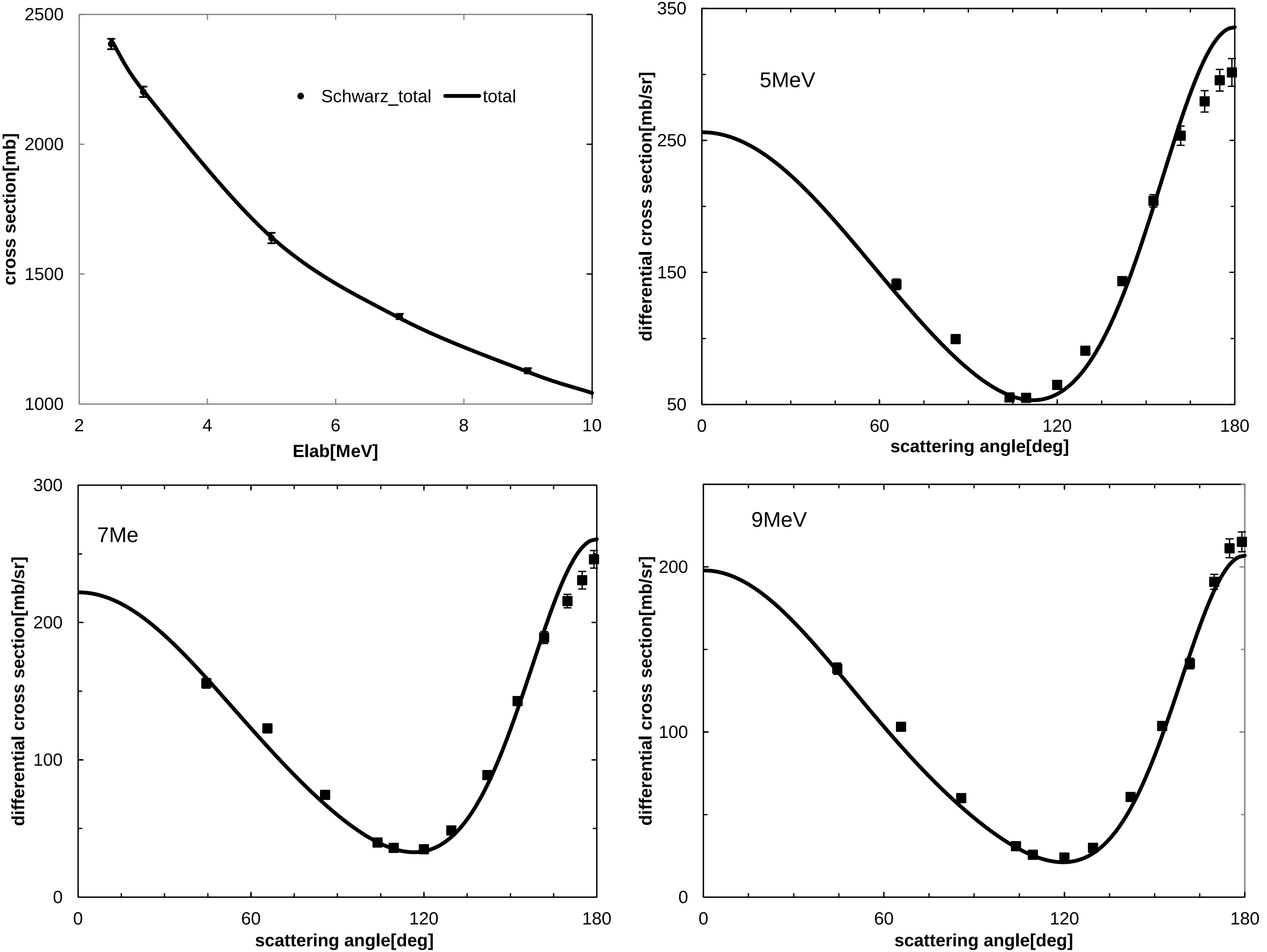


$$\begin{pmatrix} T^{\alpha} \\ T^{\beta} \\ T^{\gamma} \end{pmatrix} = \begin{pmatrix} T_{\alpha} \\ T_{\beta} \\ T_{\gamma} \end{pmatrix} + \begin{pmatrix} 0 & T_{\alpha} & T_{\alpha} \\ T_{\beta} & 0 & T_{\beta} \\ T_{\gamma} & T_{\gamma} & 0 \end{pmatrix} G_0 \begin{pmatrix} T^{\alpha} \\ T^{\beta} \\ T^{\gamma} \end{pmatrix}$$

$$T^{\alpha} = T_{\alpha} + \sum_{\beta \neq \alpha}^{3} T_{\alpha} G_0 T^{\beta}$$

$$X_{\alpha i,\beta j} = Z_{\alpha i,\beta j} + \sum_{k=1}^{K} \sum_{\gamma=1}^{3} Z_{\alpha i,\gamma k} \tau_{\gamma k} X_{\gamma k,\beta j}$$





2500
2000
1500
1000
2
4
6
8
10
cross section[mb]
Elab[MeV]
Schwarz_total
total
350
250
150
50
0
60
120
180
5MeV
differential cross section[mb/sr]
scattering angle[deg]
300
200
100
0
7Me
differential cross section[mb/sr]
scattering angle[deg]
9MeV
200
100
0
differential cross section[mb/sr]
scattering angle[deg]

Numerical results:

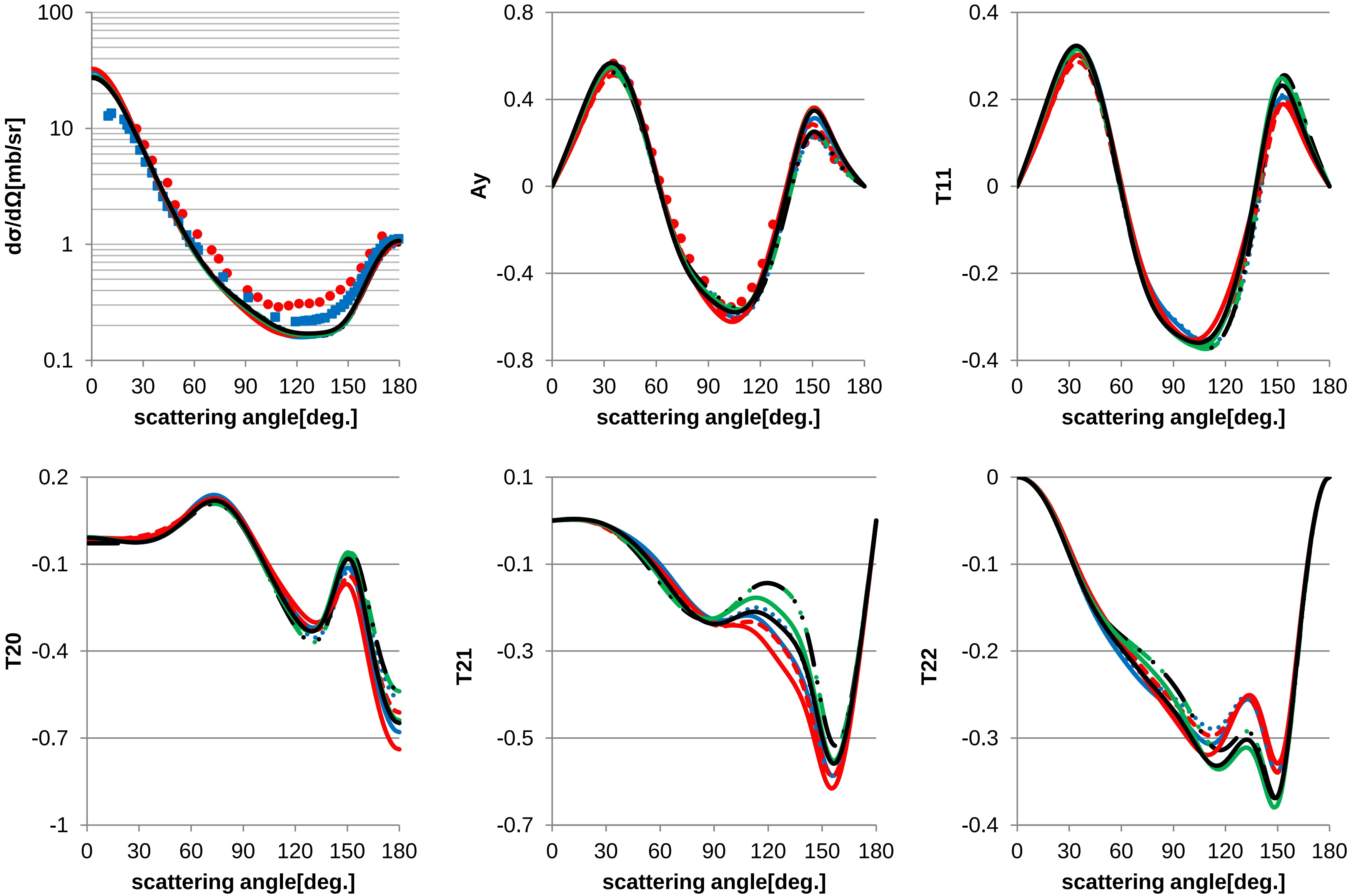
$$E_{p(lab)} = 135\text{MeV}$$

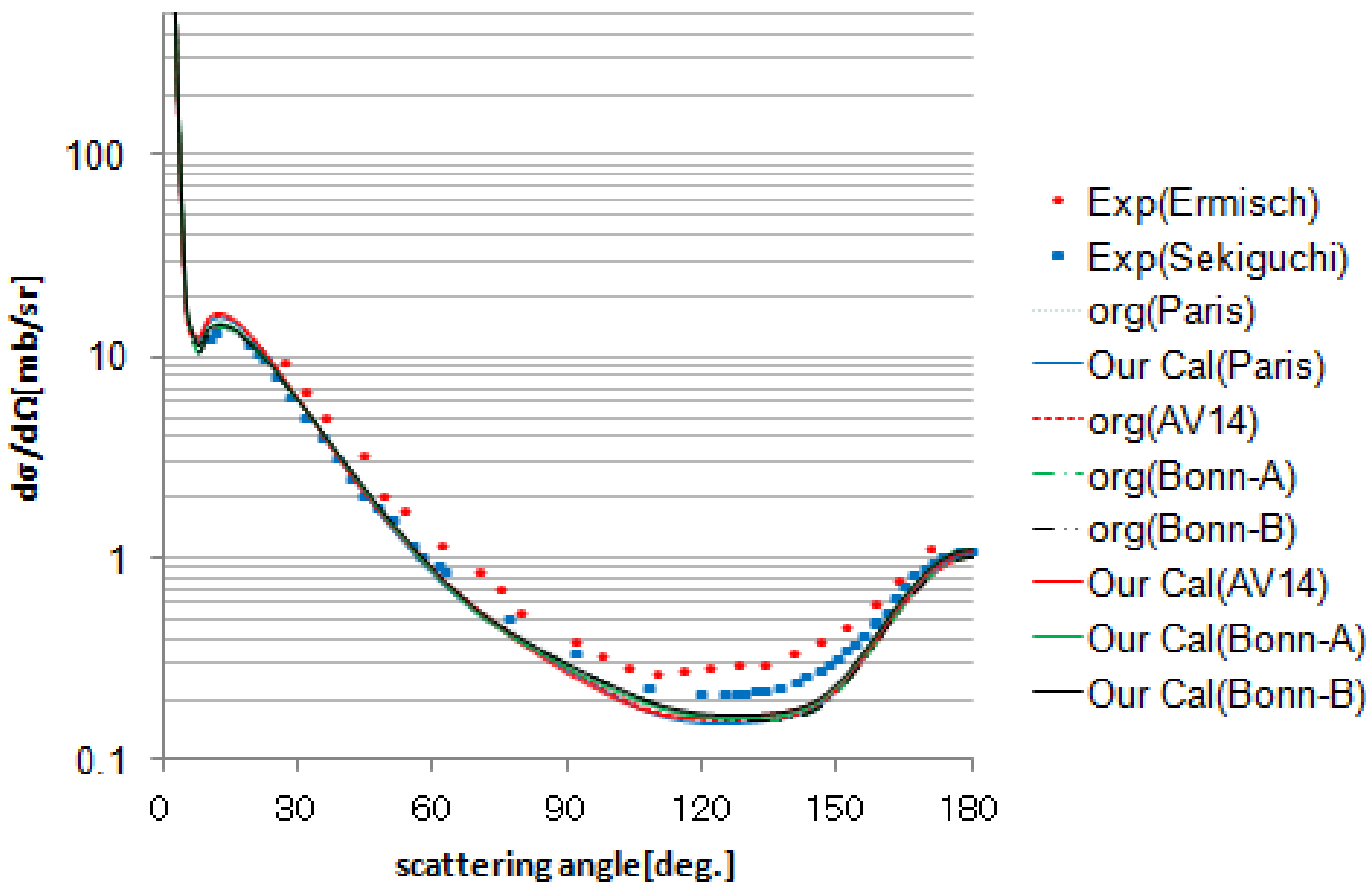
Solid curves are calculated with T_{pd}^{C}

1) blue solid : Paris,
2) red : AV14,
3) green : Bonn-A,
4) black : Bonn-B.
5) dashed curves :
   only nucleonic.

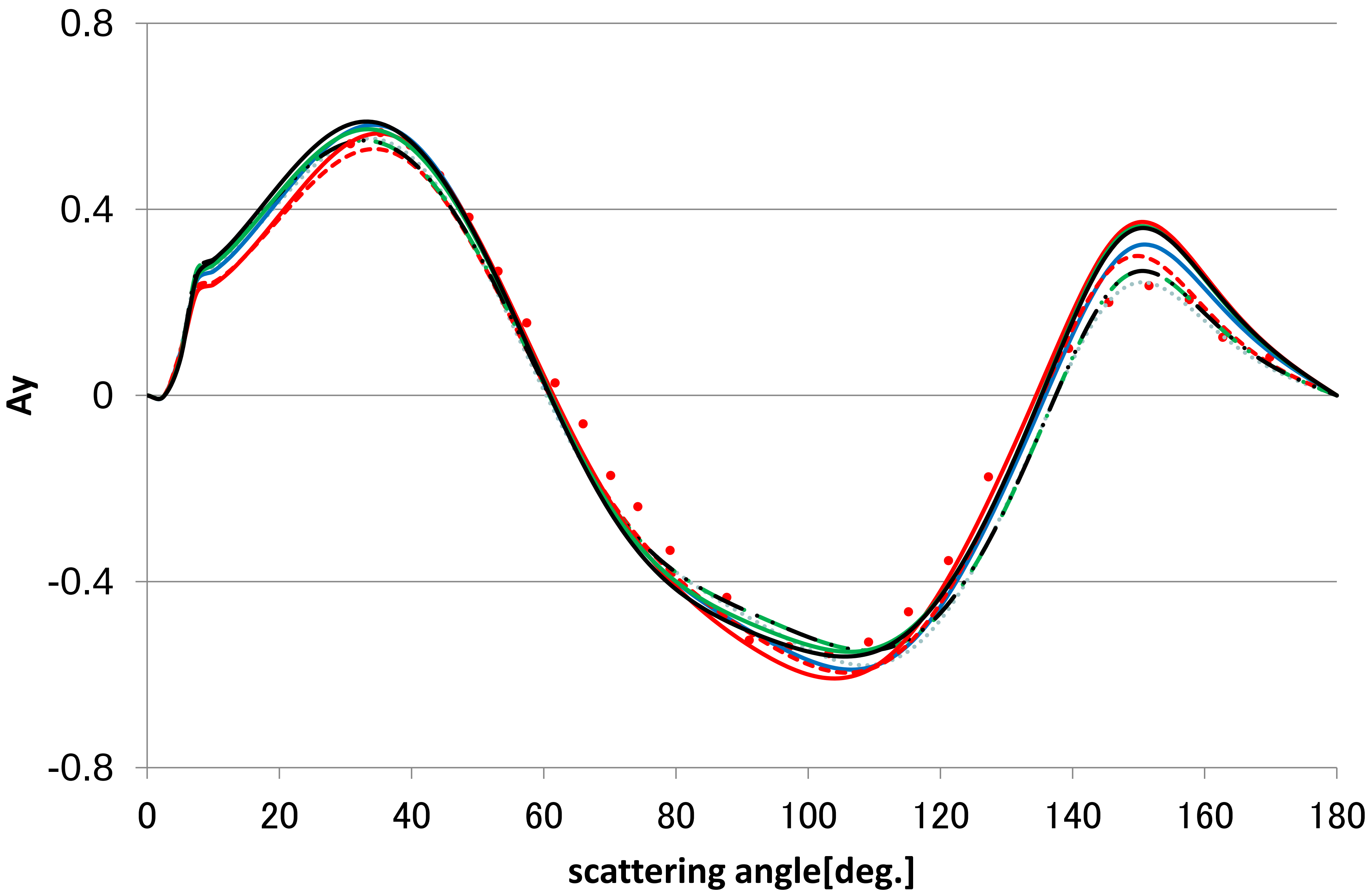
Thanks to Dr. Johan Haidenbauer for offering us these separable potentials.

$E_{p(lab)} = 135\text{MeV}$ (without T^{C}_{pd})





dσ /dΩ[mb/sr]
100
10
1
0.1
0
30
60
90
120
150
180
scattering angle[deg.]
Exp(Ermisch)
Exp(Sekiguchi)
org(Paris)
Our Cal(Paris)
org(AV14)
org(Bonn-A)
org(Bonn-B)
Our Cal(AV14)
Our Cal(Bonn-A)
Our Cal(Bonn-B)



0.8
0.4
0
-0.4
-0.8
Ay
0
20
40
60
80
100
120
140
160
180
scattering angle[deg.]

The experimental data:

1) K. Ermisch, H. R. Amir-Ahmadi, A. M. van den Berg, R.Castelijns, B.Davids et al, Phys. Rev. C 71, 064004 (2005) .

2)K. Sekiguchi, H.Sakai, H.Wital, W. Glockle, J.Golak, M. Hatano, H.Kamada, H.Kato, Y. Maeda and J. Nishikawa et al,

Phys. Rev C 65,034003 (2002)

2. A Generalization

(Multi-channel 3-body Faddeev equations)

Extension (or generalization) :

Three-body → A-body

This is an *automatic* way, *but the increase of numerical burden and the progress of hardware are always put in the balance.*

S. Oryu, S. Nemoto and P. U. Sauer, *Innovative Computational Methods in Nuclear Many-Body Problems,* edited by H. Horiuchi, M. Kamimura, H. Toki, Y. Fujiwara, M. Matsuo and Y. Sakuragi, World Scientific, (1998), 38.

To the realistic nucleus: A-body problems

1) 4-, 5- ….A-body Faddeev equations

2) Multi-channel 3-cluster Faddeev equations

The four- and many-body effects could be treated by the name of 3BF in the 3-cluster system.

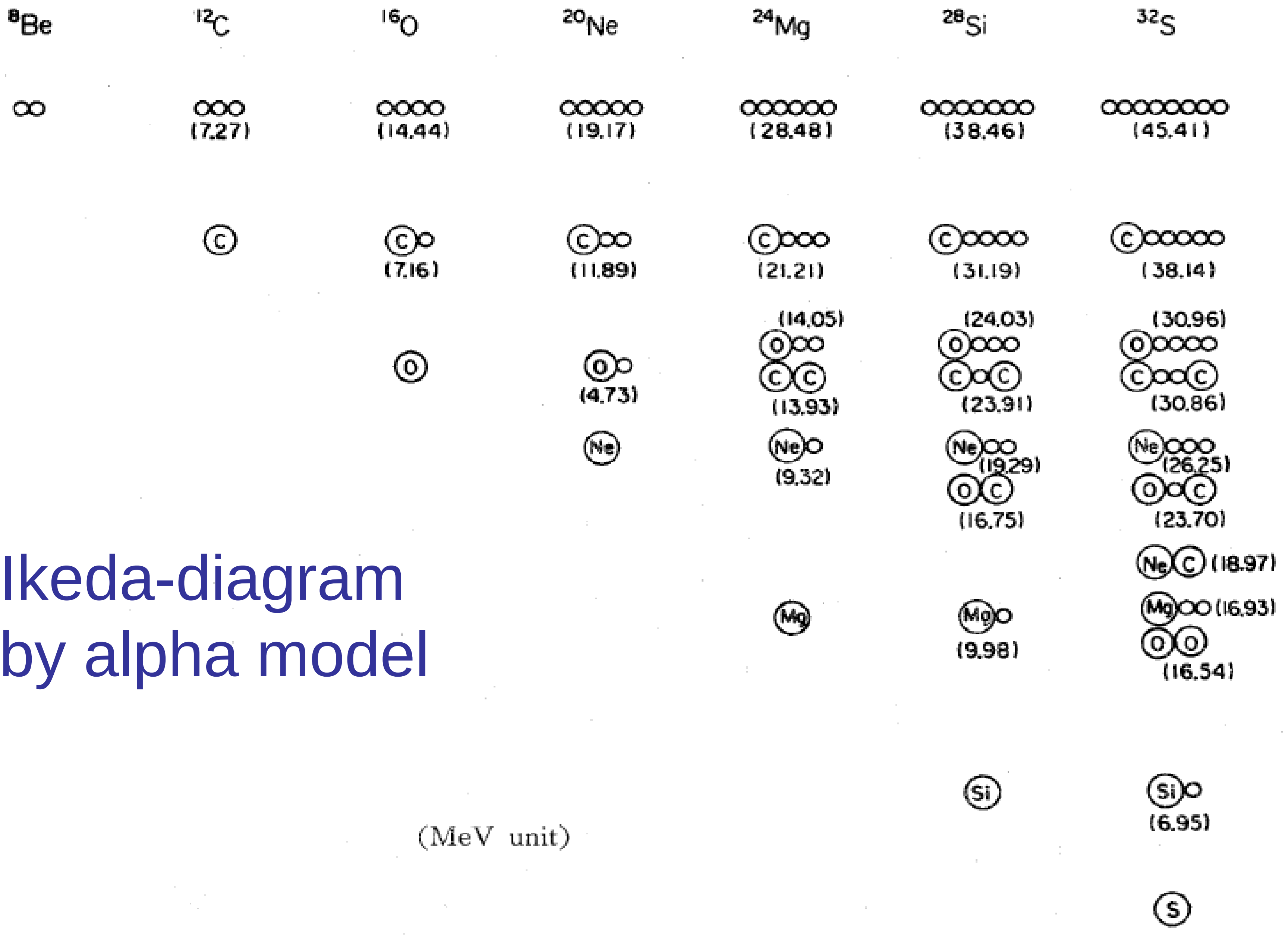
Cluster formation techniques are on the market by the well known technique:

(1) the resonationg group (RGM) technique,

(2) the orthogonal condition model (OCM),

(3) the anti-symmetric molecular dynamics (AMD),

(4) Jacobi-coordinate anti-symmetric molecular dynamics (JAMD), etc.



Ikeda-diagram by alpha model

Example

$$1)\ {}^4\mathrm{He}+\mathrm{p}\rightarrow{}^4\mathrm{He}+\mathrm{p}:\quad M=2$$

$$m=1\qquad {}^3\mathrm{He}+\mathrm{n}+\mathrm{p}$$

$$m=2\qquad {}^3\mathrm{H}+\mathrm{p}+\mathrm{p}$$

$$2)\ {}^4\mathrm{He}+\mathrm{d}\rightarrow{}^4\mathrm{He}+\mathrm{d}:\quad M=4$$

$$m=1\qquad {}^4\mathrm{He}+\mathrm{n}+\mathrm{p}$$

$$m=2\qquad {}^3\mathrm{He}+\mathrm{n}+\mathrm{d}$$

$$m=3\qquad {}^3\mathrm{H}+\mathrm{p}+\mathrm{d},$$

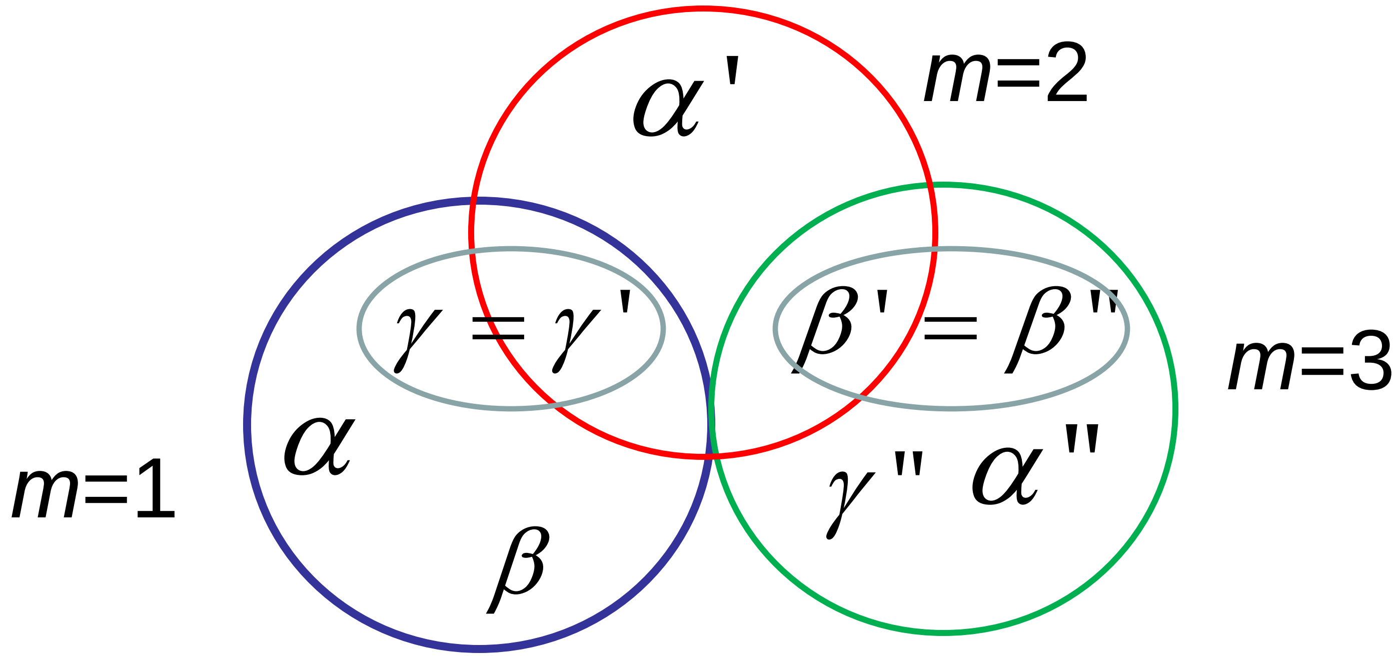
$$m=4\qquad \mathrm{d}+\mathrm{d}+\mathrm{d}$$

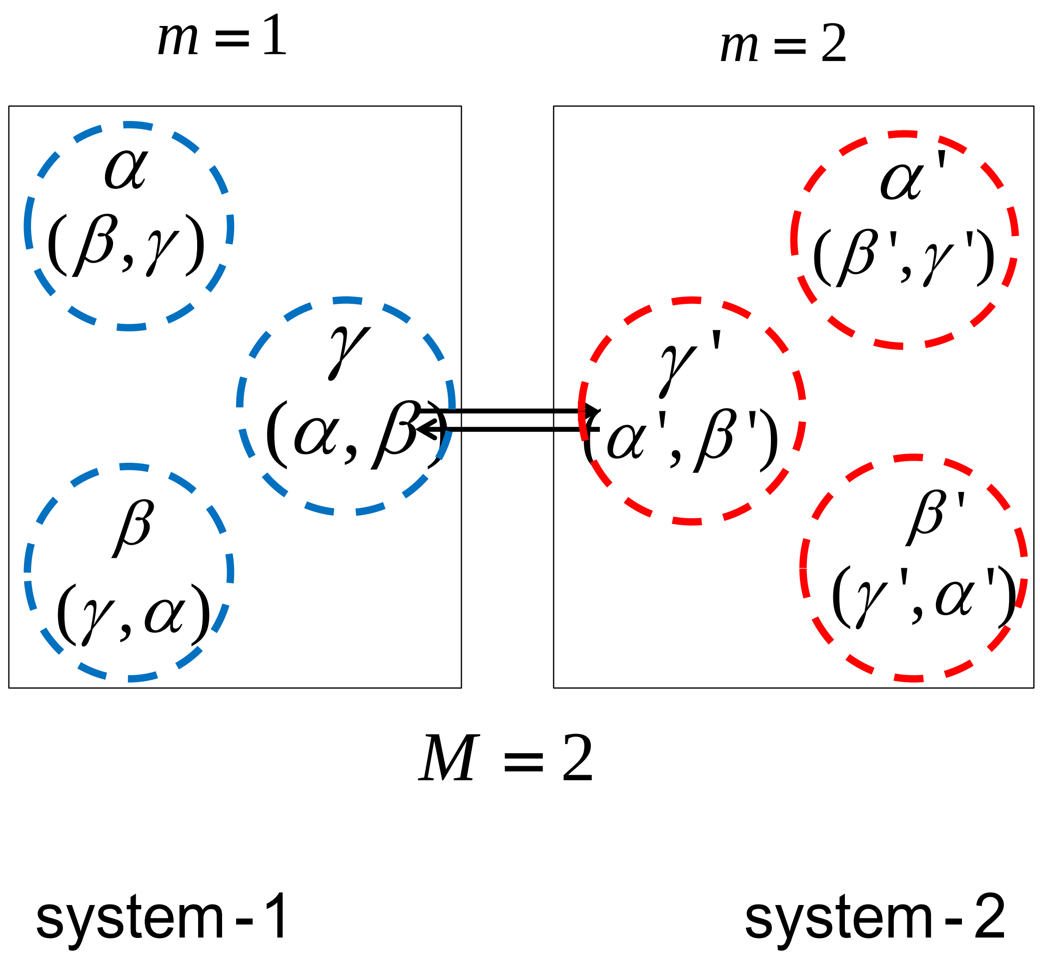
Multi-channel 3-body Faddeev equations:

Three cluster separation method

for A-body system: $M=3$

System-channel number, or multiplicity





$m=1$
$m=2$
α
(β,γ)
γ
(α,β)
β
(γ,α)
α'
(β',γ')
γ'
(α',β')
β'
(γ',α')
$M=2$
system - 1
system - 2

$$X^{a,b}_{\alpha n,\beta' m} = Z^{a,b}_{\alpha n,\beta' m} + \sum_{c,d=1}^{M}\sum_{\gamma=1}^{3}\sum_{s,t=1}^{N} Z^{a,c}_{\alpha n,\gamma s}\,\tau^{c,d}_{\gamma s,\gamma'' t}\,X^{d,b}_{\gamma'' t,\beta' m}$$

$$Z^{a,b}_{\alpha n,\beta' m} = \mathrm{g}^{a}_{\alpha n} G_0 \mathrm{g}^{b}_{\beta' m}(1-\delta_{\alpha\beta'})\delta_{ab}$$

$$\tau^{c,d}_{\gamma s,\gamma'' t} : \; n \in \alpha \in a, \;\; m \in \beta' \in b,$$

$$s \in \gamma \in c, \; t \in \gamma'' \in d$$

a, b, c, d : system numbers

$\alpha, \beta, \gamma, \delta$: channel numbers

m, n, s, t : physical states

S. Oryu, S. Nemoto and P. U. Sauer,
Innovative Computational Methods in Nuclear Many-Body Problems, edited by H. Horiuchi, M. Kamimura, H. Toki, Y. Fujiwara, M. Matsuo and Y. Sakuragi, World Scientific, (1998), 38.

a) ^{3}He+n+p and ^{3}H+p+p coupled system

$$
\begin{bmatrix}
X^{11}_{\alpha\alpha} & X^{11}_{\alpha\beta} & X^{11}_{\alpha\gamma} & X^{12}_{\alpha\alpha} & X^{12}_{\alpha\beta} & X^{12}_{\alpha\gamma} \\
X^{11}_{\beta\alpha} & X^{11}_{\beta\beta} & X^{11}_{\beta\gamma} & X^{12}_{\beta\alpha} & X^{12}_{\beta\beta} & X^{12}_{\beta\gamma} \\
X^{11}_{\gamma\alpha} & X^{11}_{\gamma\beta} & X^{11}_{\gamma\gamma} & X^{12}_{\gamma\alpha} & X^{12}_{\gamma\beta} & X^{12}_{\gamma\gamma} \\
X^{21}_{\alpha\alpha} & X^{21}_{\alpha\beta} & X^{21}_{\alpha\gamma} & X^{22}_{\alpha\alpha} & X^{22}_{\alpha\beta} & X^{22}_{\alpha\gamma} \\
X^{21}_{\beta\alpha} & X^{21}_{\beta\beta} & X^{21}_{\beta\gamma} & X^{22}_{\beta\alpha} & X^{22}_{\beta\beta} & X^{22}_{\beta\gamma} \\
X^{21}_{\gamma\alpha} & X^{21}_{\gamma\beta} & X^{21}_{\gamma\gamma} & X^{22}_{\gamma\alpha} & X^{22}_{\gamma\beta} & X^{22}_{\gamma\gamma}
\end{bmatrix}
=
\begin{bmatrix}
0 & Z^{11}_{\alpha\beta} & Z^{11}_{\alpha\gamma} & 0 & 0 \\
Z^{11}_{\beta\alpha} & 0 & Z^{11}_{\beta\gamma} & 0 & 0 \\
Z^{11}_{\gamma\alpha} & Z^{11}_{\gamma\beta} & 0 & 0 & 0 \\
0 & 0 & 0 & Z^{22}_{\alpha\beta} & Z^{22}_{\alpha\gamma} \\
0 & 0 & Z^{22}_{\beta\alpha} & 0 & Z^{22}_{\beta\gamma} \\
0 & 0 & Z^{22}_{\gamma\alpha} & Z^{22}_{\gamma\beta} & 0
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
0 & Z^{11}_{\alpha\beta} & Z^{11}_{\alpha\gamma} & 0 & 0 \\
Z^{11}_{\beta\alpha} & 0 & Z^{11}_{\beta\gamma} & 0 & 0 \\
Z^{11}_{\gamma\alpha} & Z^{11}_{\gamma\beta} & 0 & 0 & 0 \\
0 & 0 & 0 & Z^{22}_{\alpha\beta} & Z^{22}_{\alpha\gamma} \\
0 & 0 & Z^{22}_{\beta\alpha} & 0 & Z^{22}_{\beta\gamma} \\
0 & 0 & Z^{22}_{\gamma\alpha} & Z^{22}_{\gamma\beta} & 0
\end{bmatrix}
\begin{bmatrix}
\tau^{11}_{\alpha} & & & \tau^{12}_{\alpha} & & \\
& \tau^{11}_{\beta} & & & 0 & \\
& & \tau^{11}_{\gamma} & & & 0 \\
\tau^{21}_{\alpha} & & & \tau^{22}_{\alpha} & & \\
& 0 & & & \tau^{22}_{\beta} & \\
& & 0 & & & \tau^{22}_{\gamma}
\end{bmatrix}
\begin{bmatrix}
X^{11}_{\alpha\alpha} & X^{11}_{\alpha\beta} & X^{11}_{\alpha\gamma} & X^{12}_{\alpha\alpha} & X^{12}_{\alpha\beta} & X^{12}_{\alpha\gamma} \\
X^{11}_{\beta\alpha} & X^{11}_{\beta\beta} & X^{11}_{\beta\gamma} & X^{12}_{\beta\alpha} & X^{12}_{\beta\beta} & X^{12}_{\beta\gamma} \\
X^{11}_{\gamma\alpha} & X^{11}_{\gamma\beta} & X^{11}_{\gamma\gamma} & X^{12}_{\gamma\alpha} & X^{12}_{\gamma\beta} & X^{12}_{\gamma\gamma} \\
X^{21}_{\alpha\alpha} & X^{21}_{\alpha\beta} & X^{21}_{\alpha\gamma} & X^{22}_{\alpha\alpha} & X^{22}_{\alpha\beta} & X^{22}_{\alpha\gamma} \\
X^{21}_{\beta\alpha} & X^{21}_{\beta\beta} & X^{21}_{\beta\gamma} & X^{22}_{\beta\alpha} & X^{22}_{\beta\beta} & X^{22}_{\beta\gamma} \\
X^{21}_{\gamma\alpha} & X^{21}_{\gamma\beta} & X^{21}_{\gamma\gamma} & X^{22}_{\gamma\alpha} & X^{22}_{\gamma\beta} & X^{22}_{\gamma\gamma}
\end{bmatrix}
\tag{24}
$$

Here, potential elements are of two types:

$$
Z^{11}_{\alpha\beta}(q,q';E) = \langle g^1_\alpha(p)|\, G^{(1)}_0(E)\, |g^1_\beta(p')\rangle \overline{\delta_{\alpha\beta}},
$$

and

$$
Z^{22}_{\alpha\beta}(q,q';E) = \langle g^2_\alpha(p)|\, G^{(2)}_0(E)\, |g^2_\beta(p')\rangle \overline{\delta_{\alpha\beta}}. \tag{25}
$$

b) ^{4}He-n-p, ^{3}He-n-d, ^{3}H-p-d, and d-d-d systems (^{6}Li-nucleus)[7]

$$
\begin{array}{llll}
m=1 & \alpha:\alpha_1(n_2p_3), & \beta:n_2(p_3\alpha_1), & \gamma:p_3(\alpha_1n_2) \\
m=2 & \alpha:h_1(n_2d_3), & \beta:n_2(d_3h_1), & \gamma:d_3(h_1n_2) \\
m=3 & \alpha:t_1(p_2d_3), & \beta:p_2(d_3t_1), & \gamma:d_3(t_1p_2) \\
m=4 & \alpha:d_1(d_2d_3), & \beta:d_2(d_3d_1), & \gamma:d_3(d_1d_2)
\end{array}
$$

$$
\tau = \begin{bmatrix}
\tau_\alpha^{11} & & & 0 & & & 0 & & & 0 & & \\
& \tau_\beta^{11} & & & \tau_\beta^{12} & & & \tau_\beta^{13} & & & 0 & \\
& & \tau_\gamma^{11} & & & 0 & & & 0 & & & 0 \\
0 & & & \tau_\alpha^{22} & & & 0 & & & \tau_\alpha^{24} & & \\
& \tau_\beta^{21} & & & \tau_\beta^{22} & & & 0 & & & \tau_\beta^{24} & \\
& & 0 & & & \tau_\gamma^{22} & & & \tau_\gamma^{23} & & & \tau_\gamma^{24} \\
0 & & & 0 & & & \tau_\alpha^{33} & & & \tau_\alpha^{34} & & \\
& \tau_\beta^{31} & & & 0 & & & \tau_\beta^{33} & & & \tau_\beta^{34} & \\
& & 0 & & & \tau_\gamma^{32} & & & \tau_\gamma^{33} & & & \tau_\gamma^{34} \\
0 & & & \tau_\alpha^{42} & & & \tau_\alpha^{43} & & & \tau_\alpha^{44} & & \\
& 0 & & & \tau_\beta^{42} & & & \tau_\beta^{43} & & & \tau_\beta^{44} & \\
& & 0 & & & \tau_\gamma^{42} & & & \tau_\gamma^{43} & & & \tau_\gamma^{44}
\end{bmatrix} \tag{29}
$$

It should be noted that the matrix is not equal to the MTCC one in which τ_β^{31} and τ_β^{13} were missed, even if one neglects the d-d-d partition.[7]

c) Three-nucleon system coupled with Δ-isobar resonances

$$
\begin{array}{ll}
m=1 & \alpha{:}N_1(N_2N_3),\ \ \beta{:}N_2(N_3N_1),\ \ \gamma{:}N_3(N_1N_2) \\
m=2 & \alpha{:}\Delta_1(N_2N_3),\ \ \beta{:}N_2(N_3\Delta_1),\ \ \gamma{:}N_3(\Delta_1N_2) \\
m=3 & \alpha{:}\Delta_1(\Delta_2N_3),\ \ \beta{:}\Delta_2(N_3\Delta_1),\ \ \gamma{:}N_3(\Delta_1\Delta_2) \\
m=4 & \alpha{:}\Delta_1(\Delta_2\Delta_3),\ \ \beta{:}\Delta_2(\Delta_3\Delta_1),\ \ \gamma{:}\Delta_3(\Delta_1\Delta_2).
\end{array}
$$

$$
\boldsymbol{\tau} = \begin{bmatrix}
\tau_\alpha^{11} & & & 0 & & & 0 & & & 0 & & \\
& \tau_\beta^{11} & & & \tau_\beta^{12} & & & 0 & & & 0 & \\
& & \tau_\gamma^{11} & & & \tau_\gamma^{12} & & & \tau_\gamma^{13} & & & 0 \\
0 & & & \tau_\alpha^{22} & & & \tau_\alpha^{23} & & & \tau_\alpha^{24} & & \\
& \tau_\beta^{21} & & & \tau_\beta^{22} & & & \tau_\beta^{23} & & & \tau_\beta^{24} & \\
& & \tau_\gamma^{21} & & & \tau_\gamma^{22} & & & \tau_\gamma^{23} & & & \tau_\gamma^{24} \\
0 & & & \tau_\alpha^{32} & & & \tau_\alpha^{33} & & & \tau_\alpha^{34} & & \\
& 0 & & & \tau_\beta^{32} & & & \tau_\beta^{33} & & & \tau_\beta^{34} & \\
& & \tau_\gamma^{31} & & & \tau_\gamma^{32} & & & \tau_\gamma^{33} & & & \tau_\gamma^{34} \\
0 & & & \tau_\alpha^{42} & & & \tau_\alpha^{43} & & & \tau_\alpha^{44} & & \\
& 0 & & & \tau_\beta^{42} & & & \tau_\beta^{43} & & & \tau_\beta^{44} & \\
& & 0 & & & \tau_\gamma^{42} & & & \tau_\gamma^{43} & & & \tau_\gamma^{44}
\end{bmatrix}.
$$

Multi-Channel 3-Body Faddeev Equations(MC3F)

Merit:

1) directly connect to the 3-body Faddeev equations.
2) Multiplicity is only mixed with the two-body propagators and without double counting.
3) Time and memory saving for one program run.

Demerit:

1) Burden for the preparation of the inter-cluster interactions.
2) Numerical burden between A-body equations with nuclear potential and MC3F with inter-cluster potentials is always put in the balance.

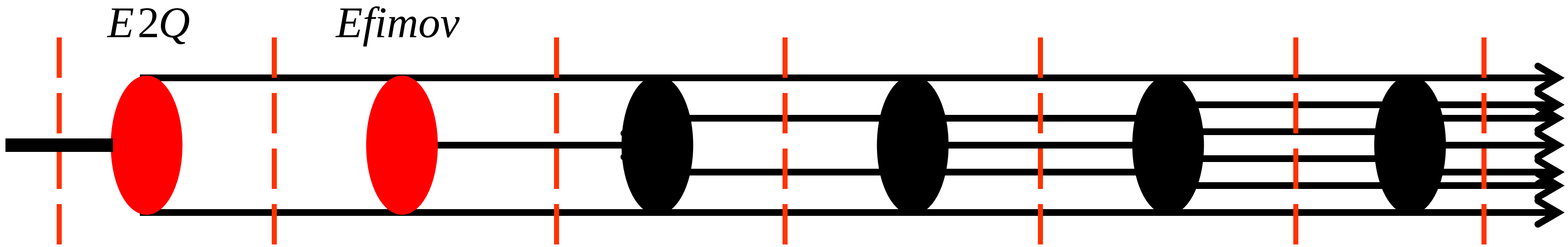
Note: MTCC by Miyagawa et al. (1986) is similar to our MC3F, but the Born terms and the kernels may be different.

K. Miyagawa, T. Ueda, T. Sawada and S. Takagi, Nucl. Phys. **A459** (1986) 93.

3) Reduction:

Three-body → Two-body

the multi-channel Lippmann-Schwinger (MLS) equations below the 3-body break up threshold is constructed, where the 3-body Faddeev equations are analytically continued to the MLS equations.

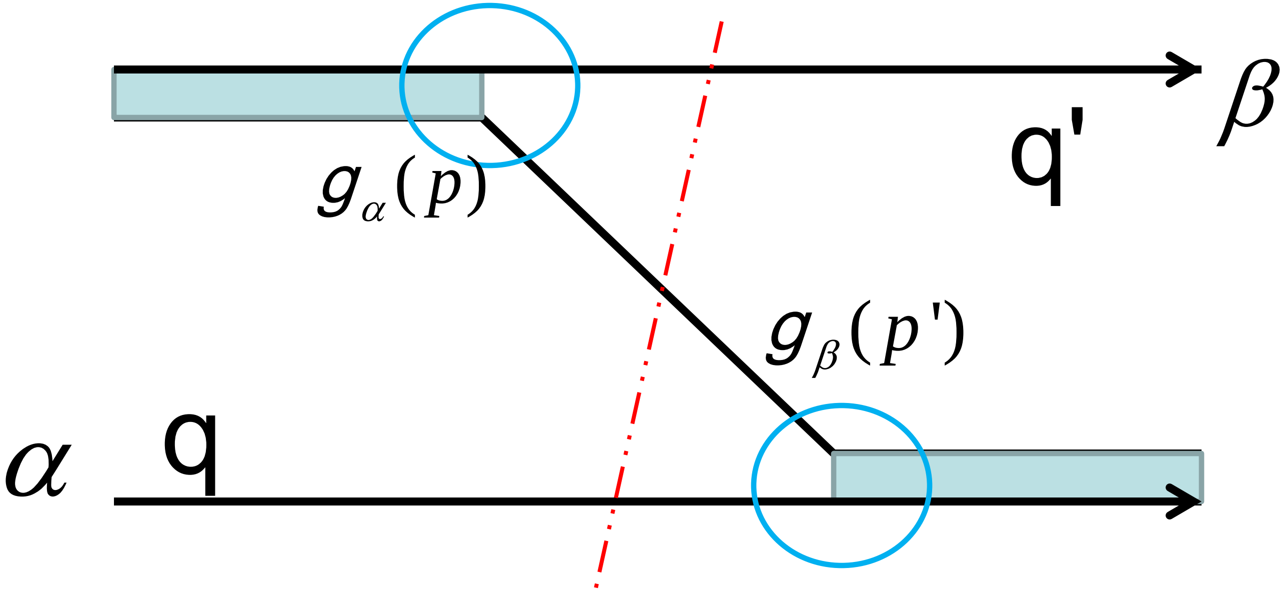


$$\begin{pmatrix} H_0^{(1)} \\ E^{(1)} \end{pmatrix} \Leftarrow \begin{pmatrix} H_0^{(2)} \\ E^{(2)} \end{pmatrix} \Leftarrow \begin{pmatrix} H_0^{(3)} \\ E^{(3)} \end{pmatrix} \Leftarrow \begin{pmatrix} H_0^{(4)} \\ E^{(4)} \end{pmatrix} \Leftarrow \begin{pmatrix} H_0^{(5)} \\ E^{(5)} \end{pmatrix} \Leftarrow \begin{pmatrix} H_0^{(6)} \\ E^{(6)} \end{pmatrix} \Leftarrow \begin{pmatrix} H_0^{(7)} \\ E^{(7)} \end{pmatrix}$$

$$\Updownarrow \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow$$

$$\begin{pmatrix} \overline{H}_0^{(2)} \\ E_{cm}^{(2)} \end{pmatrix} \Leftarrow \begin{pmatrix} \overline{H}_0^{(3)} \\ E_{cm}^{(3)} \end{pmatrix} \Leftarrow \begin{pmatrix} \overline{H}_0^{(4)} \\ E_{cm}^{(4)} \end{pmatrix} \Leftarrow \begin{pmatrix} \overline{H}_0^{(5)} \\ E_{cm}^{(5)} \end{pmatrix} \Leftarrow \begin{pmatrix} \overline{H}_0^{(6)} \\ E_{cm}^{(6)} \end{pmatrix} \Leftarrow \begin{pmatrix} \overline{H}_0^{(7)} \\ E_{cm}^{(7)} \end{pmatrix}$$

In the Lovelace's idea in early1960s, the 2 - body Hamiltonian : $H_0^{(2)}$ is represented by a virtual 3 - body Hamiltonian: $\overline{H}_0^{(3)}$, where the potential is the energy dependent 2 - body quasi - potential (E2Q) which has a singularity at the threshold.



Potential

$$Z_{\alpha\beta}(\mathrm{q},\mathrm{q}';E) = \frac{g_\alpha(p)g_\beta(p')}{D(\mathrm{q},\mathrm{q}';E)}$$

$$D_{\mathrm{Fadd}}(\mathrm{q},\mathrm{q}';E) \equiv \sqrt{S} - \omega_1(\mathrm{q}_1) - \omega_2(\mathrm{q}_2) - \omega_3(\mathrm{q}_3)$$

$$= (\sqrt{S} + m) - (\omega_1(\mathrm{q}_1) + \omega_\beta(\mathrm{q}_\beta) + \omega_\gamma(\mathrm{q}_\gamma) + m)$$

$$= (\sqrt{S} + m) - (\omega_1(\bar{\mathrm{q}}_1) + \omega_\beta(\bar{\mathrm{q}}_\beta) + \omega_\gamma(\bar{\mathrm{q}}_\gamma)) \equiv D_{\mathrm{E2Q}}(\mathrm{q},\mathrm{q}';E)$$

$$D_{\text{Fadd}}(\mathrm{q},\mathrm{q}';E) \equiv \left(\sqrt{S}-2M-m\right)-\left(\omega_1(\mathrm{q}_1)+\omega_2(\mathrm{q}_2)+\omega_3(\mathrm{q}_3)-2M-m\right)$$

$$\approx E-\frac{q_1^{\,2}}{2M}-\frac{q_2^{\,2}}{2M}-\frac{q_3^{\,2}}{2m}=E-\frac{q_{1,2}^{\,2}}{2\mu_{,21}}-z_{1,2}=\boxed{E-H_0} \qquad \text{(A)}$$

$$D_{\text{E2Q}}(\mathrm{q},\mathrm{q}';E) \equiv \left(\sqrt{S}+m-2M-m\right)-\left(\omega_1(\bar{\mathrm{q}}_1)+\omega_2(\bar{\mathrm{q}}_2)+\omega_3(\bar{\mathrm{q}}_3)-2M-m\right)$$

$$\approx (E+m)-\frac{\bar{q}_1^{\,2}}{2M}-\frac{\bar{q}_2^{\,2}}{2M}-\frac{\bar{q}_3^{\,2}}{2m}=E_{\text{cm}}-\frac{\bar{q}_{1,2}^{\,2}}{2\mu_{1,2}}-\bar{z}_{1,2}=\boxed{E_{\text{cm}}-\bar{H}_0} \qquad \text{(B)}$$

$$\boxed{\frac{\bar{q}_{1,2}^{\,2}}{2\mu_{1,2}}=\frac{q_{1,2}^{\,2}}{2\mu_{1,2}}+m}, \qquad \text{(C)}$$

Substituting (C) to (B), and comparing (A), $\therefore\ \boxed{\bar{z}_{1,2}=z_{1,2}}$ (D)

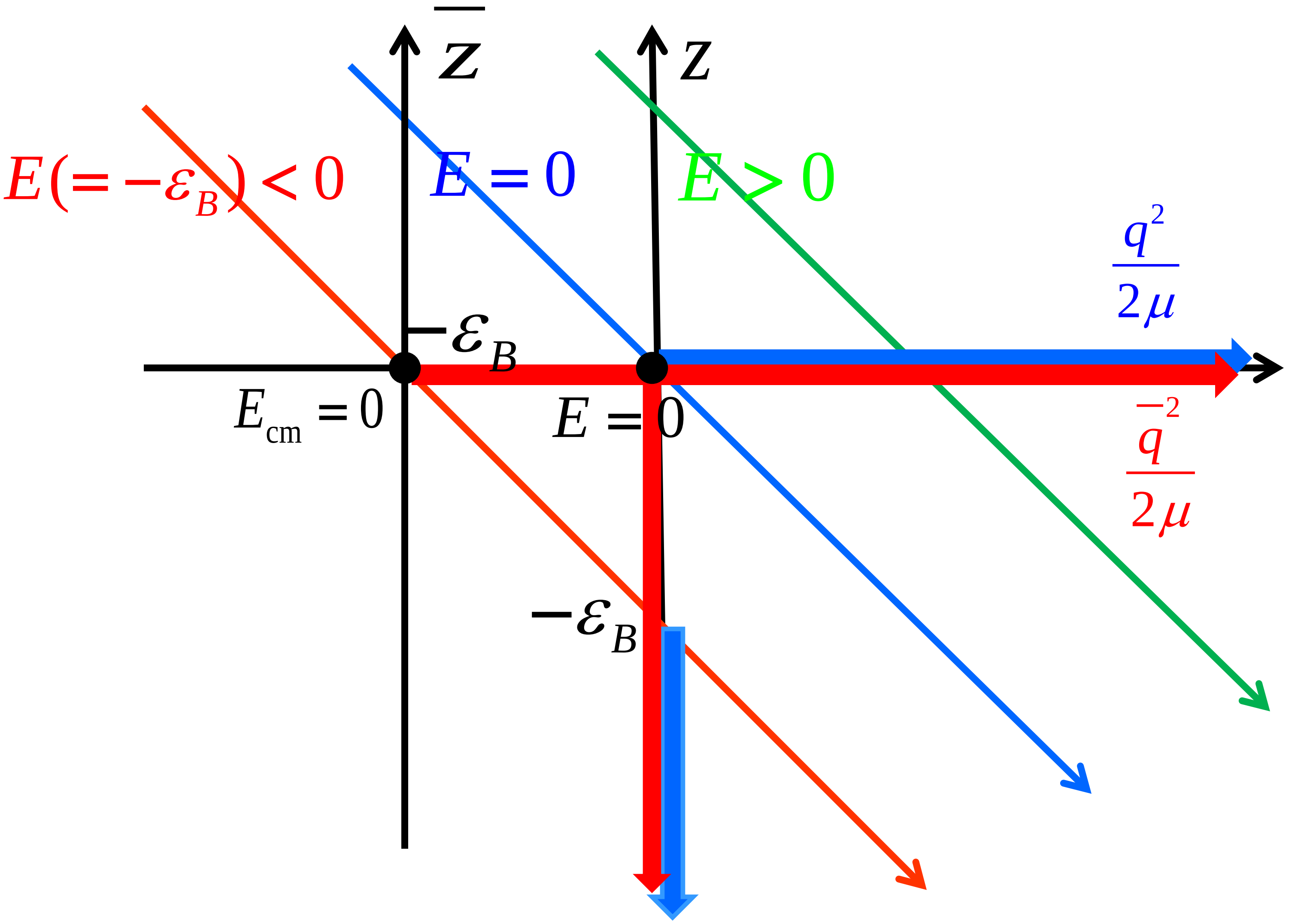
Two-body energy doesn't change!

$D_{\text{Fadd}}(\mathrm{q},\mathrm{q}';E)=D_{\text{E2Q}}(\mathrm{q},\mathrm{q}';E)$, however Hamiltonian changes,

so that integral variable changes!

Therefore, two-body informations are different, i.e., original Faddeev method is missing lower energy informations.

$$z = E - \frac{q^2}{2\mu} = \left(E + \varepsilon_B\right) - \left(\frac{q^2}{2\mu} + \varepsilon_B\right) \equiv E_{\text{cm}} - \frac{\bar{q}^2}{2\mu} = \bar{z}$$



$$\frac{\bar{q}_{1,2}^{\,2}}{2\mu_{1,2}} = \frac{q_{1,2}^{\,2}}{2\mu_{1,2}} + m \,, \qquad \therefore \ \bar{z}_{1,2} = z_{1,2}$$

$$\frac{\bar{q}_{3}^{\,2}}{2\mu_{3}} = \frac{q_{3}^{\,2}}{2\mu_{3}} + \alpha \,, \qquad (\alpha \text{ :unknown})$$

$$D_{\mathrm{E2Q}}(\mathrm{q},\mathrm{q}';E) \equiv \left(\sqrt{S} + \alpha - 2M - m\right) - \left(\omega_1(\bar{\mathrm{q}}_1) + \omega_2(\bar{\mathrm{q}}_2) + \omega_3(\bar{\mathrm{q}}_3) - 2M - m\right)$$

$$\approx (E+\alpha) - \frac{\bar{q}_1^{\,2}}{2M} - \frac{\bar{q}_2^{\,2}}{2M} - \frac{\bar{q}_3^{\,2}}{2m} = E_{\mathrm{cm}} - \frac{\bar{q}_3^{\,2}}{2\mu_{1,2}} - \bar{z}_3 = E_{\mathrm{cm}} - \bar{H}_0$$

$$= (E+\alpha) - \left(\frac{q_3^{\,2}}{2\mu_{1,2}} - \alpha\right) - \bar{z}_3 = E - \frac{q_3^{\,2}}{2\mu_{1,2}} - \bar{z}_3 \qquad \therefore \ \bar{z}_3 = z_3$$

2 - body sub - energy doesn't change by E2Q transformation.

Summining up $q_i^2 / 2m_i$ with respect to $i = 1, 2, 3$

$$\frac{\bar{q}_1^2}{2m_1} + \frac{\bar{q}_2^2}{2m_2} + \frac{\bar{q}_3^2}{2m_3} = \left(\frac{q_1^2}{2m_1} + \frac{q_2^2}{2m_2} + \frac{q_3^2}{2m_3} \right) + \left(\frac{2\mu_1 m}{2m_1} + \frac{2\mu_2 m}{2m_2} + \frac{2\mu_3}{2m_3}\alpha \right),$$

$$\therefore \left(\frac{2\mu_1 m}{2m_1} + \frac{2\mu_2 m}{2m_2} + \frac{2\mu_3}{2m_3}\alpha \right) = m$$

$$\alpha = \frac{(2M+m)}{2M}\left(m - \frac{2(M+m)m}{(2M+m)} \right) = \frac{-m^2}{2M}$$

$$\boxed{\frac{\bar{q}_3^{\,2}}{2\mu_3} = \frac{q_3^{\,2}}{2\mu_3} - \frac{m^2}{2M}},$$

$\bar{q}_3$ virtual for $\bar{q}_3^2 < 0$

and $$\frac{\bar{q}_3^{\,2}}{2m_3} = \frac{q_3^{\,2}}{2m_3} - \frac{\mu_3 m^2}{2m_3 M} = \frac{q_3^{\,2}}{2m_3} - \frac{m^2}{(2M+m)}$$

Difference occurs

1) at NN' threshold: E2Q: $E_{cm} = 0,$

a) Integral variable: $0 \le \bar{q}_{1,2,3} < \infty \qquad \frac{\bar{q}_{1,2}^{\,2}}{2\mu_{1,2}} = \frac{q_{1,2}^{\,2}}{2\mu_{1,2}} + m\,,$

b) Denominator: $[D_{E2Q}]^{-1} = \left[E_{\rm cm} - \frac{\bar{q}_{1,2}^{\,2}}{2\mu_{1,2}} - \bar{z}_{1,2} \right]^{-1}$

$$\Rightarrow \left[-\frac{\bar{q}_1^{\,2}}{2m_1} - \frac{\bar{q}_2^{\,2}}{2m_2} - \frac{(\bar{\rm q}_1 + \bar{\rm q}_2)^2}{2m_3} \right]^{-1}$$ has a singuler logarithmic cut

at NN' threshold: Faddeev: $E = -m,$

a) Integral variable: $0 \le q_{1,2,3} < \infty$

b) Denominator: $[D_{Fadd}]^{-1} = \left[E - \frac{q_{1,2}^{\,2}}{2\mu_{1,2}} - z_{1,2} \right]^{-1} \Rightarrow \left[-m - \frac{q_{1,2}^{\,2}}{2\mu_{1,2}} - z_{1,2} \right]^{-1}$

$$\Rightarrow \left[-m - \frac{q_1^{\,2}}{2m_1} - \frac{q_2^{\,2}}{2m_2} - \frac{({\rm q}_1 + {\rm q}_2)^2}{2m_3} \right]^{-1}$$ is a regular function

2) Second difference is a missing region:

$$\frac{\bar{q}_{1,2}^{\,2}}{2\mu_{1,2}} = \frac{q_{1,2}^{\,2}}{2\mu_{1,2}} + m\,, \qquad \therefore\ \bar{q}_{1,2}^{\,2} = q_{1,2}^2 + 2\mu_{1,2}m$$

$$\frac{\bar{q}_{3}^{\,2}}{2\mu_{3}} = \frac{q_{3}^{\,2}}{2\mu_{3}} + \alpha \qquad \therefore\ \bar{q}_{3}^{\,2} = q_3^2 + 2\mu_3\alpha = q_3^2 - \mu_3\frac{m^2}{M}$$

$$0 \le \bar{q}_{1,2}^{\,2} \le 2\mu_{1,2}m = \frac{2M(M+m)}{2M+m}m \approx Mm$$

gives $-2\mu_{1,2}m \le q_{1,2}^2 \le 0$: this is a missing region

$0 \le \bar{q}_3^{\,2}$ corresponds to $\mu_3\frac{m^2}{M} \le q_3^2$;

$$0 \le q_3^2 \le \mu_3\frac{m^2}{M} = \frac{m \times 2M}{2M+m}\,\frac{m^2}{M} = \frac{2m^3}{2M+m} \approx \frac{m^3}{M} << Mm$$

is missing in the $\bar{q}_3^{\,2}$ integral, but very small.

3) A phenomenon at the 3-body break up threshold : *E=0*

The Efimov Effect

Accumulated 3-Body bound states

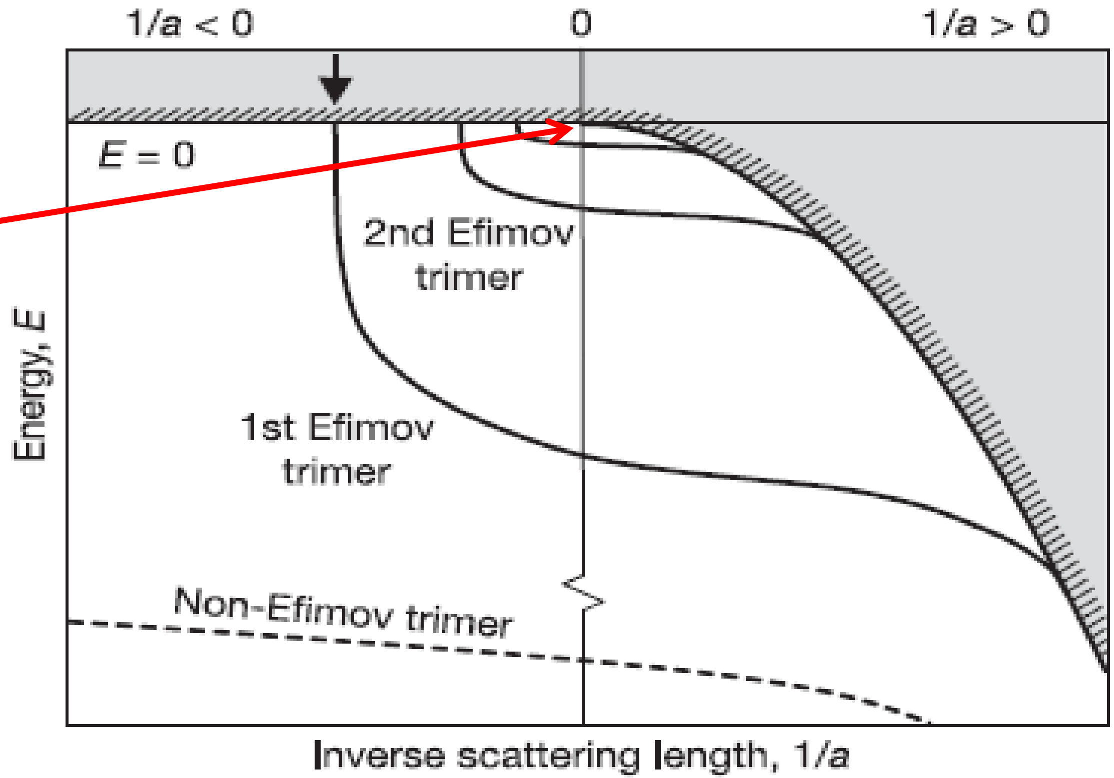


Figure 1 | Efimov's scenario. Appearance of an infinite series of weakly bound Efimov trimer states for resonant two-body interaction. The binding energy is plotted as a function of the inverse two-body scattering length $1/a$. The shaded region indicates the scattering continuum for three atoms ($a < 0$) and for an atom and a dimer ($a > 0$). The arrow marks the intersection of the first Efimov trimer with the three-atom threshold. To illustrate the series of Efimov states, we have artificially reduced the universal scaling factor from 22.7 to 2. For comparison, the dashed line indicates a tightly bound non-Efimov trimer[30], which does not interact with the scattering continuum.

Difference occurs

1) at NN' threshold: E2Q: $E_{cm} = 0,$

a) Integral variable: $0 \le \bar{q}_{1,2,3} < \infty$ $\quad \frac{\bar{q}_{1,2}{}^2}{2\mu_{1,2}} = \frac{q_{1,2}{}^2}{2\mu_{1,2}} + m$,

b) Denominator: $[D_{E2Q}]^{-1} = \left[E_{\rm cm} - \frac{\bar{q}_{1,2}{}^2}{2\mu_{1,2}} - \bar{z}_{1,2} \right]^{-1}$

$$\Rightarrow \left[-\frac{\bar{q}_1{}^2}{2m_1} - \frac{\bar{q}_2{}^2}{2m_2} - \frac{(\bar{q}_1 + \bar{q}_2)^2}{2m_3} \right]^{-1}$$ has a singular logarithmic cut

At the 3-body threshold $E = 0$

~~at NN' threshold: Faddeev: $E = -m,$~~

a) Integral variable: $0 \le q_{1,2,3} < \infty$

b) Denominator: $[D_{Fadd}]^{-1} = \left[E - \frac{q_{1,2}{}^2}{2\mu_{1,2}} - z_{1,2} \right]^{-1} \Rightarrow \left[0 - \frac{q_{1,2}{}^2}{2\mu_{1,2}} - z_{1,2} \right]^{-1}$

$$\Rightarrow \left[-\frac{q_1{}^2}{2m_1} - \frac{q_2{}^2}{2m_2} - \frac{(q_1 + q_2)^2}{2m_3} \right]^{-1}$$ has a singular logarithmic cut.

Recent development :

Efimov effect (1970) is experimentally found in atomic system by Kraemer et al. (2006).

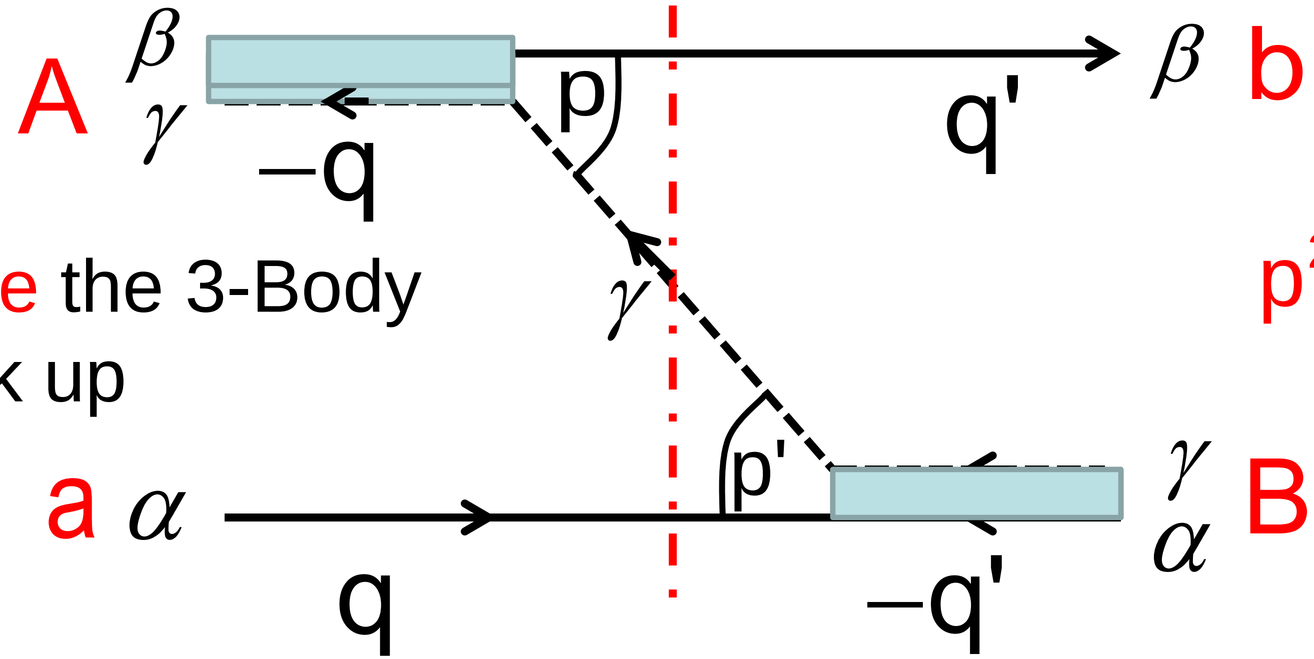
V. Efimov, Phys. Lett. 33B 563~564 (1970) .
Energy levels arising from resonant two-body forces in a three-body system,

Kraemer, T. et al. Nature 440 315-318 (2006)
Evidence for Efimov quantum states in an ultracold gas of caesium atoms,

4) In a fourth difference from the original Faddeev,

a phenomenon below the 3-body threshold emerges as a long range NN' (or [N-(Nπ)]) in the 3-body NNπ system.

S. Oryu, Phys. Rev**. C86**. 044001-1-10 (2012);
ibid. Few-Body Syst**. 54**, 1-4, 283-286 (2013).



Above the 3-Body Break up

$p^2 > 0$

Faddeev Born

$$Z_{\alpha n,\beta m}(-\mathbf{q},\mathbf{q}';E) = \frac{-g_{\alpha n}(\mathbf{p})\, m_\gamma\, g_{\beta m}(\mathbf{p}')\,\bar{\delta}_{\alpha,\beta}}{E - \left(\dfrac{\mathbf{q}^2}{2m_\alpha} + \dfrac{\mathbf{q}'^2}{2m_\beta} + \dfrac{(\mathbf{q}-\mathbf{q}')^2}{2m_\gamma} \right)}$$

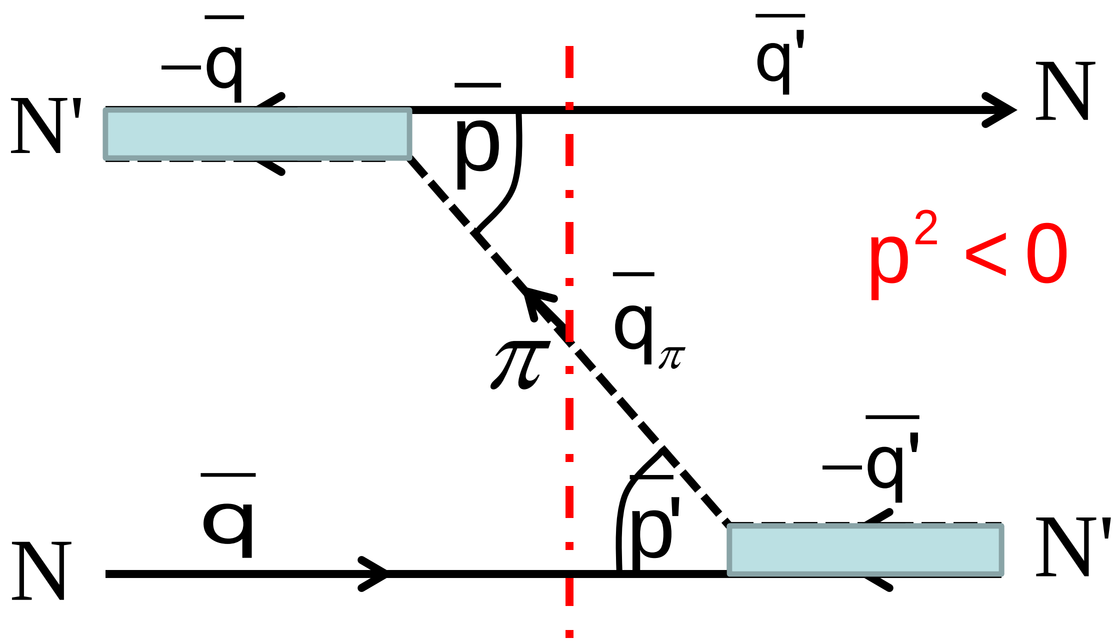
Let me show you our reduction from 3-body to 2-body equations.

$$Z_{n,m}(-\mathrm{q},\mathrm{q}';E) = \frac{-\mathrm{g}_n(\mathrm{p})m_\pi \mathrm{g}_m(\mathrm{p}')}{(E+m_\pi) - \left(\frac{\mathrm{q}^2}{2M}+\frac{\mathrm{q}'^2}{2M}+\frac{(\mathrm{q}-\mathrm{q}')^2}{2m_\pi}+m_\pi\right)}$$

$NN\pi - system$

Below the 3-Body Break up

$$= \frac{-\mathrm{g}_n(\mathrm{p})m_\pi \mathrm{g}_m(\mathrm{p}')}{\overline{E} - \left(\frac{\overline{\mathrm{q}}^2}{2M}+\frac{\overline{\mathrm{q}}'^2}{2M}+\frac{(\overline{\mathrm{q}}-\overline{\mathrm{q}}')^2}{2m_\pi}\right)}$$



Hereafter Let us use momentum without bar for simplicity

$\overline{\mathrm{q}} \to \mathrm{q}$

$\overline{\mathrm{q}}' \to \mathrm{q}'$

3) E2Q (Energy Dependent 2-Body Quasi) Potential

$$Z_{\alpha n,\beta m}(-\mathrm{q},\mathrm{q}';E)=\frac{-g_{\alpha n}(\mathrm{p})m_\pi g_{\beta m}(\mathrm{p}')\overline{\delta}_{\alpha,\beta}}{qq'(\chi\Lambda-x)}\equiv\frac{C_{\alpha n,\beta m}(\mathrm{p},\mathrm{p}')}{qq'(\chi\Lambda-x)}$$

with $\Lambda=1+{m_\pi}/{M}=1+\Delta=1+0.147$

$$\chi\Lambda=\frac{-2m_\pi\overline{E}+\Lambda q^2+\Lambda q'^2}{2qq'}\to\frac{\sigma^2+q^2+q'^2}{2qq'}\Lambda$$

$$x={\mathbf{qq'}}/{qq'}\,;\qquad \chi=\frac{\sigma^2+q^2+q'^2}{2qq'}\,;$$

$$\frac{-2m_\pi\overline{E}}{\Lambda}=\frac{2m_\pi(|E|-m_\pi)}{\Lambda}\equiv\sigma^2>0$$

Bound state case

I) 2-body threshold: 3-body free energy: $E=-|E|$,

$$\frac{-2m_\gamma \bar{E}}{\Lambda}=\frac{-2m_\gamma(E+|\varepsilon_B|)}{\Lambda}=\frac{2m_\gamma(|E|-|\varepsilon_B|)}{\Lambda}\equiv \boxed{\sigma^2=0}$$

a) NN'-bound state: $\bar{E}=-|E|+m_\pi\le 0 \quad (or\ \ 0<\sigma^2)$

b) NN'-scat. length cal.: $0\le\bar{E}=-|E|+m_\pi \ \ (\sigma^2<0)$

NN' threshold: $|E|=m_\pi$

c) π+d scat. length cal.: $0\le\bar{E}=-|E|+\varepsilon_d \ \ (\sigma^2<0)$

πd threshold: $|E|=\varepsilon_d$

II) 3-body threshold: $(|E|-|\varepsilon_B|)=-|\varepsilon_B|\equiv\frac{\Lambda}{2m_\gamma}\sigma^2$

if adopt: $\sigma^2=0$, then $\varepsilon_B=0$ (or $a\to\pm\infty$)

I) 2-body threshold: 3-body free energy: $E = -|E|$,

$$\frac{-2m_\gamma \bar{E}}{\Lambda} = \frac{-2m_\gamma (E + |\varepsilon_B|)}{\Lambda} = \frac{2m_\gamma (|E| - |\varepsilon_B|)}{\Lambda} \equiv \sigma^2 = 0$$

a) NN'-bound state: $\bar{E} = -|E| + m_\pi \leq 0 \quad (or \;\; 0 < \sigma^2)$

b) NN'-scat. length cal.: $0 \leq \bar{E} = -|E| + m_\pi \;\; (\sigma^2 < 0)$

NN' threshold: $|E| = m_\pi$

c) π+d scat. length cal.: $0 \leq \bar{E} = -|E| + \varepsilon_d \;\; (\sigma^2 < 0)$

πd threshold: $|E| = \varepsilon_d$

II) 3-body threshold: $(|E| - |\varepsilon_B|) = -|\varepsilon_B| \equiv \frac{\Lambda}{2m_\gamma}\sigma^2$

if adopt: $\sigma^2 = 0$, then $\varepsilon_B = 0$ (or $a \to \pm\infty$)

I) 2-body threshold: 3-body free energy: $E=-|E|$,

$$\frac{-2m_\gamma \overline{E}}{\Lambda}=\frac{-2m_\gamma(E+|\varepsilon_B|)}{\Lambda}=\frac{2m_\gamma(|E|-|\varepsilon_B|)}{\Lambda}\equiv \boxed{\sigma^2=0}$$

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πd threshold: $|E|=\varepsilon_d$

II) 3-body threshold: $(|E|-|\varepsilon_B|)=-|\varepsilon_B|\equiv \frac{\Lambda}{2m_\gamma}\sigma^2$

if adopt: $\sigma^2=0,$ then $\varepsilon_B=0$ (or $a\to\pm\infty$)

Efimov case !

$$\Lambda = 1 + \Delta : \qquad (\Delta \equiv m_\pi / M = 0.147);$$

$$(\Lambda\chi - x)^{-1} = (\chi + \Delta\chi - x)^{-1}$$

Δ expansion of Green's function,

$$Z_{\alpha n,\beta m}(\text{-q, q'}; E)$$

$$= 2C_{\alpha n,\beta m}(\text{p,p'})\sum_{j=0}^{\infty}(-\Delta)^j \frac{(\sigma^2 + q^2 + q'^2)^j}{[\sigma^2 + (\text{q} - \text{q'})^2]^{j+1}}$$

Two-body potential with energy dependence.

Fourier tranceform; with $C_{\alpha n,\beta m}(\text{p,p'}) \approx C_{\alpha n,\beta m}$

$$\mathcal{F}\left[Z_{\alpha n,\beta m}(\text{-q, q'}; E)\right] = \frac{\delta(\text{R})C_{\alpha n,\beta m}}{4\pi(2+\Delta)}U(\Delta,\sigma;r)$$

$$U(\Delta,\sigma;r)=\frac{1}{r}e^{-\sigma r/2}+\left(\frac{\Delta}{2+\Delta}\right)\frac{(-1)}{1!2^2}\sigma e^{-\sigma r/2}$$

$$+\left(\frac{\Delta}{2+\Delta}\right)^2\frac{(-1)^2(\sigma r/2+1)}{2!2^3}\sigma e^{-\sigma r/2}$$

$$+\left(\frac{\Delta}{2+\Delta}\right)^3\frac{(-1)^3(\sigma^2 r^2/2+3\sigma r+6)}{3!2^5}\sigma e^{-\sigma r/2}$$

$$+\ .....$$

$$\equiv U^{(0)}(\Delta,\sigma;r)+U^{(1)}(\Delta,\sigma;r)+U^{(2)}(\Delta,\sigma;r)+....$$

The two-body potential reduction has the energy dependence.

we adopt a statistical average with a weight:

$$P = \frac{\sigma^{2\gamma+1} e^{-a\sigma}}{\rho}$$

$$\rho = \int_0^\infty \sigma^{2\gamma+1} e^{-a\sigma} d\sigma = \frac{\Gamma(2\gamma+2)}{a^{2\gamma+2}}$$

$$\mathcal{L}\left\{U^{(0)}(\Delta,\sigma;r)\right\} \equiv \frac{1}{\rho}\int_0^\infty \sigma^{2\gamma+1} e^{-a\sigma} \frac{e^{-\sigma r/2}}{r} d\sigma$$

$$= \frac{a^{2\gamma+2}}{r(r/2+a)^{2\gamma+2}}$$

Weight function $\sigma^{2\gamma+1}e^{-a\sigma}/\rho$ denotes

nucleon structure (or form factor) effects.

1) Van der Waals type: $\sigma^{2\gamma+1}e^{-a\sigma} \to \sigma^{4}e^{-a\sigma}$

by $\gamma = 3/2$ and for $2a \equiv a_0$

$$\mathcal{L}\left\{U^{(0)}(\Delta,\sigma;r)\right\} = \frac{a_0^5}{r(r+a_0)^5}$$

$$\to \frac{e^{-5r/a_0}}{r} \quad \text{for } r \ll a_0$$

$$\to \frac{a_0^5}{r^6} \quad \text{for } r \gg a_0$$

2) Monotonic: $\sigma^{2\gamma+1}e^{-a\sigma} \to 1 e^{-a\sigma}$ by $\gamma = -1/2$

$$\mathcal{L}\left\{U^{(0)}(\Delta,\sigma;r)\right\} = \frac{a_0}{r(r+a_0)} \quad (\text{with } 2a = a_0)$$

$$\to \frac{e^{-\mu_0 r}}{r} \quad (\text{for } a_0 \gg r \text{ with } \mu_0 = 1/a_0)$$

$$\to \frac{a_0}{r^2} \quad (\text{for } a_0 \ll r)$$ Long range

3) Yukawa potential: $\sigma^{2\gamma+1}e^{-a\sigma} \to \delta(\sigma - 2\mu_0)$

$$\mathcal{L}\left\{U^{(0)}(\Delta,\sigma;r)\right\} = \frac{e^{-\mu_0 r}}{r}$$

Numerical calculation by Schroedinger equation:

n	E_n (MeV)	E_n/E_{n-1}	$\langle r_n^2 \rangle^{1/2}$ (fm)	$\langle r_n^2 \rangle^{1/2}/\langle r_{n-1}^2 \rangle^{1/2}$
1	-2.222		2.516	
2	-1.271×10^{-2}	174.8	3.652×10^{1}	14.52
3	-7.433×10^{-5}	171.0	4.812×10^{2}	13.18
4	-4.347×10^{-7}	171.0	6.296×10^{3}	13.08
5	-2.543×10^{-9}	171.0	8.233×10^{4}	13.08
6	-1.487×10^{-11}	171.0	1.077×10^{6}	13.08
7	-8.697×10^{-14}	171.0	1.408×10^{7}	13.08
8	-5.087×10^{-16}	171.0	1.841×10^{8}	13.08
9	-2.975×10^{-18}	171.0	2.407×10^{9}	13.08
10	-1.740×10^{-20}	171.0	3.147×10^{10}	13.08

Our analytic prediction fits to the numerical solution.

Calculated Results

For πD, NN' scattering lengths

Y. Hiratsuka, S. Oryu, and T. Watanabe,
Proc. Of the 6th APFB Conf. Adelaide 2014).

πD scattering length by our calculation using original Faddeev & E2Q

	Scattering length [fm]
Our cal. By original Faddeev (type A-potential; P_{33} resonance)	0.033
Our cal. By original Faddeev (type B-pot.; S_{11}, P_{11}, P_{33} resonance P_{11} bound state)	–0.019 +0.019*i*
E2Q (type B-pot.; S_{11}, P_{11}, P_{33} resonance P_{11} bound state)	–0.023 +0.019*i*
EXP	–0.038 +0.009*i* –0.038 +0.008*i*

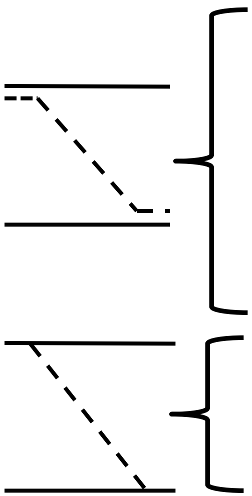
Faddeev

Faddeev

E2Q

P. Hauser et al., Phys. Rev. C58, R1869 (1998);
D. Chatellard et al., Nucl. Phys. A625, 855 (1997).

neutron-proton triplet scattering length by
Our cal. original Faddeev, & by E2Q



	Scattering length [fm]
Our cal. by Faddeev NN' (type A-pot.)	0.280
Our cal.by Faddeev NN' (type B-pot.; S_{11}, P_{11} resonance P_{11} bound state)	2.85
Our cal. by E2Q NN' (type B; S_{11}, P_{11} resonance P_{11} bound state)	4.66
EXP: for NN	5.419±0.007

T. L. Houk, PRC3, 1886 (1971); W. Dilg, PRC11,103 (1975); S. Klarsfeld et al., JPG10, 165 (1984)

Below the three-body break up threshold in NNπ system :   N+(Nπ)   or   N+N'

**Kinematical possibility is added below the three-body break up threshold,
because the nucleon variables are changed by the pion mass absorption.**

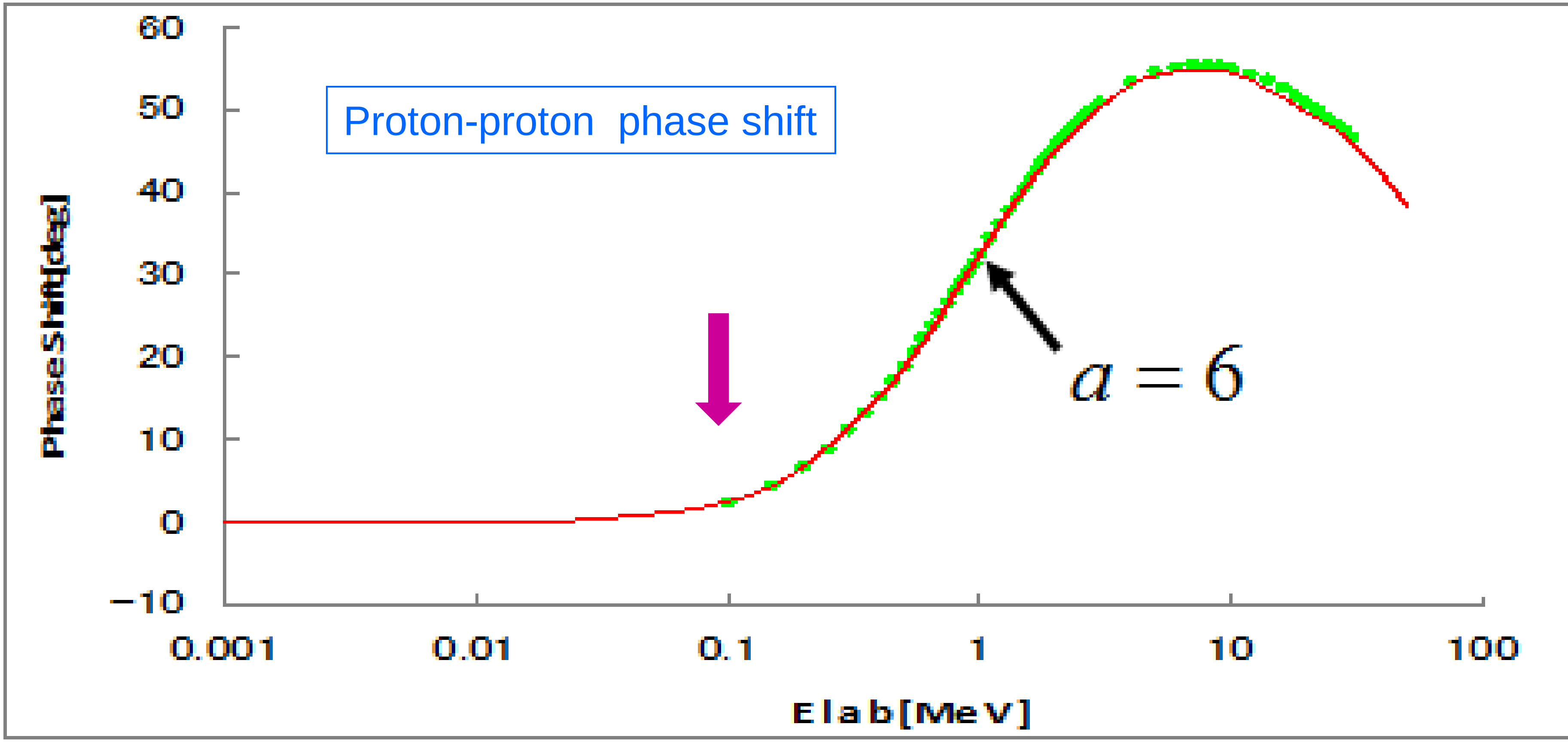
Determine the screened Coulomb range parameter;

$$R_{c\ell} = \exp(a\gamma)/2k$$

$$T^{(R)} = (V^S + V^R) + (V^S + V^R) G_0 T^{(R)},$$

$$T^R = V^R + V^R G_0 T^R,$$

$$\delta = \tan^{-1}\left(\frac{\mathrm{Im}(T^{(R)} - T^R)}{\mathrm{Re}(T^{(R)} - T^R)}\right)$$



J. R. Bergervoet, P. C. van Campen, W. A. van der Sanden, and J. J. de Swart, Phys. Rev. C38, 15 (1988)

The future aspects

1) Are there long range cluster - cluster interactions?

$$\frac{m_\pi}{M_N} \approx 0.145 \qquad N + (N, \pi) \text{ scattering}$$

$$\frac{M_N}{M_{^7Li}} \approx \frac{1}{7} = 0.143 \qquad {}^7Li + ({}^6Li, n) \text{ scattering}$$

$$\frac{M_\alpha}{M_{^{28}Si}} \approx \frac{1}{7} = 0.143 \qquad {}^{28}Si + ({}^{24}Mg, \alpha) \text{ scattering}$$

$$^{28}Si + (^{28}Si, \alpha), \ ^{32}S \text{ scatterings}$$

2) Are there neuclear E2Q energy levels?

3) Are there long range effects in unstable nucleus?

4) Are there long range effects in neutron rich nucleus?

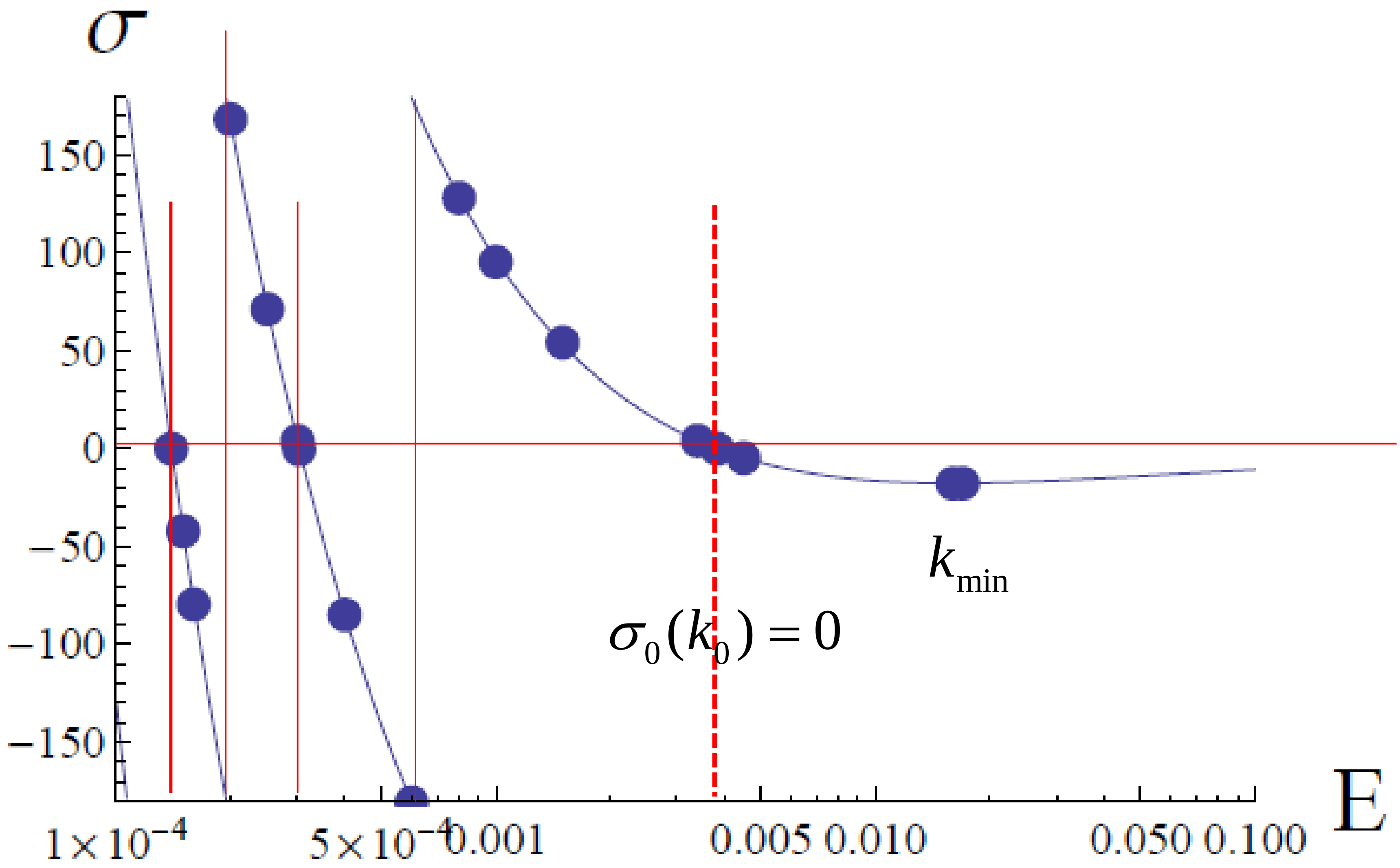
4) Recent development :

a) Research of the threshold behavior by the Faddeev's approach makes an offer a new frontier.

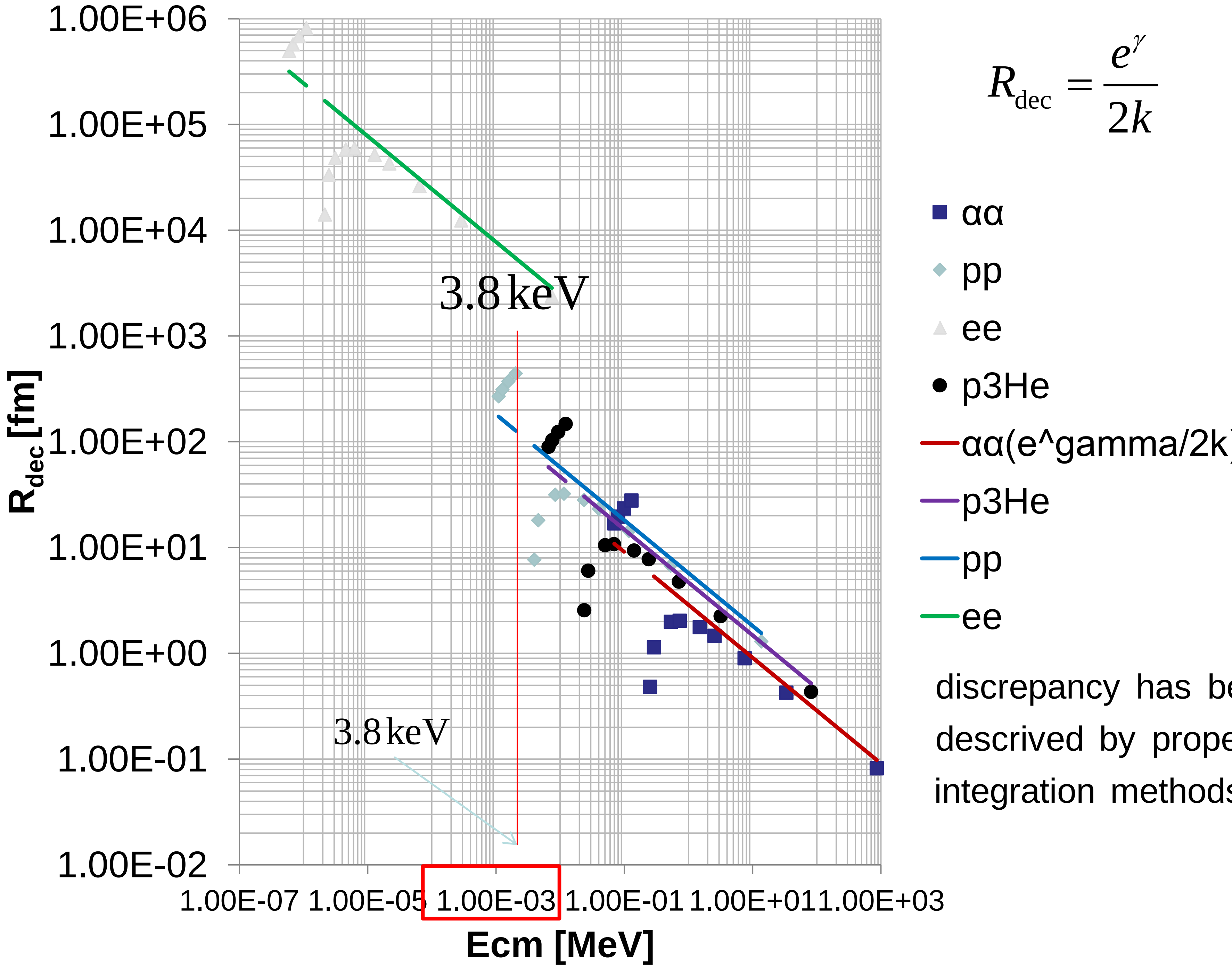
b) The Coulomb interaction is now treated in the Faddeev equations.

S. Oryu, Phys. Rev. **C73**, 054001 (2006),
*ibid***,** **C76**, 069901 (2007).
S. Oryu, Y. Hiratsuka, S. Nishinohara, S. Chiba,
J. Phys. G: Nucl. Part. Phys. **39** 045101 (2012);
ibid. Phys. Rev**. C75**, 021001 (2007).

Coulomb phase shift

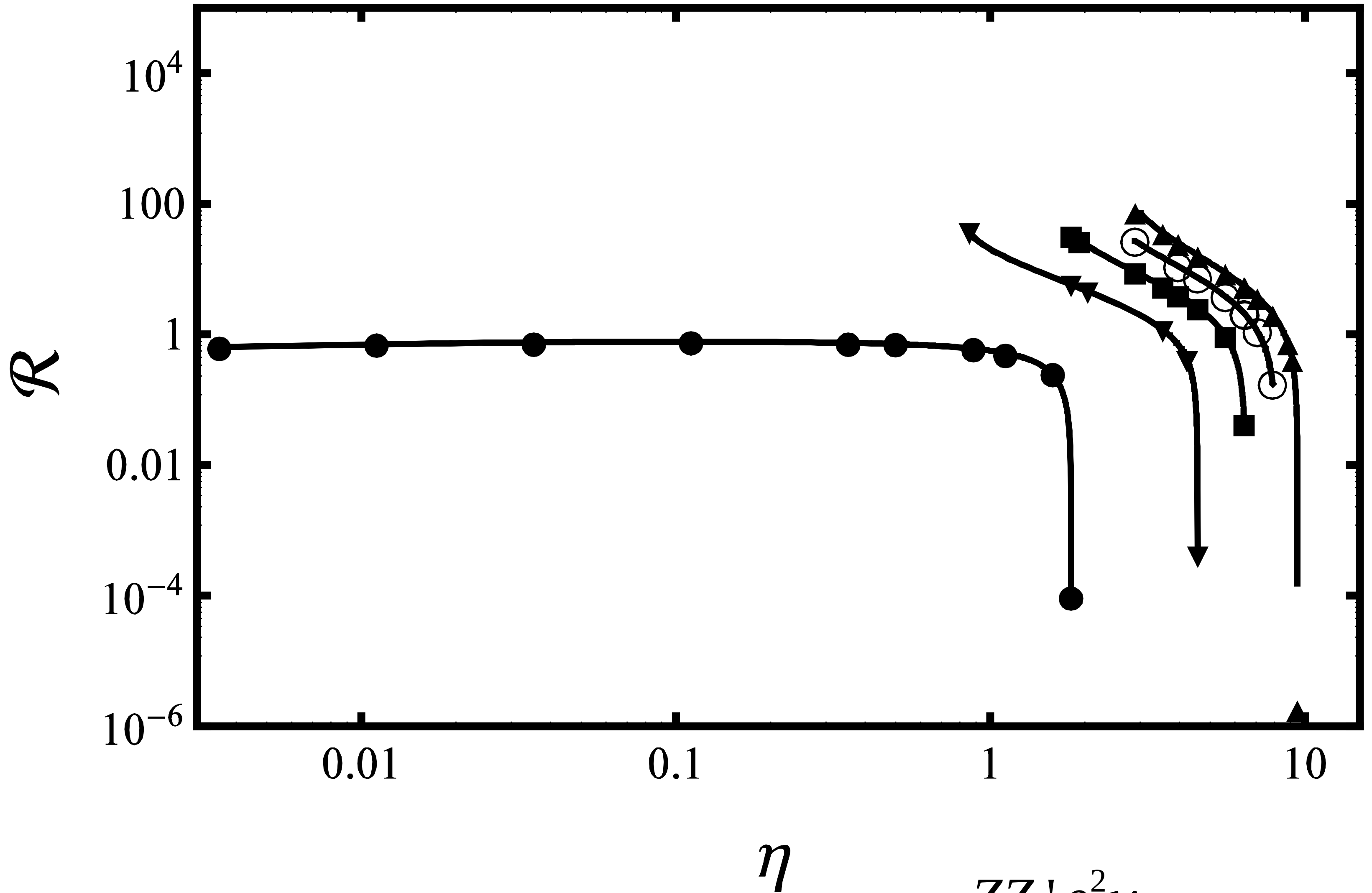


$$R_{\text{dec}} = \frac{e^{\gamma}}{2k}$$



discrepancy has been described by proper integration methods.

(kR – universal range)



$$\eta = \frac{ZZ' e^2 \nu}{k}$$

Screening ranges

$$R_0 = \frac{37.283 + 5708.9\eta - 3166.9\eta^2}{64.616 + 7062.0\eta - 2564.0\eta^2}$$

$$R_1 = \frac{-259.88 + 447.042\eta - 142.14\eta^2 + 12.414\eta^3}{10.301 - 38.589\eta + 36.964\eta^2 - 5.8202\eta^3}$$

$$R_2 = \frac{42.289 - 42.199\eta + 26.9977\eta^2 - 3.3228\eta^3}{0.050814 + 0.32437\eta - 0.91936\eta^2 + 0.57813\eta^3}$$

$$R_3 = \ldots\ldots\ldots$$

Universal variables:

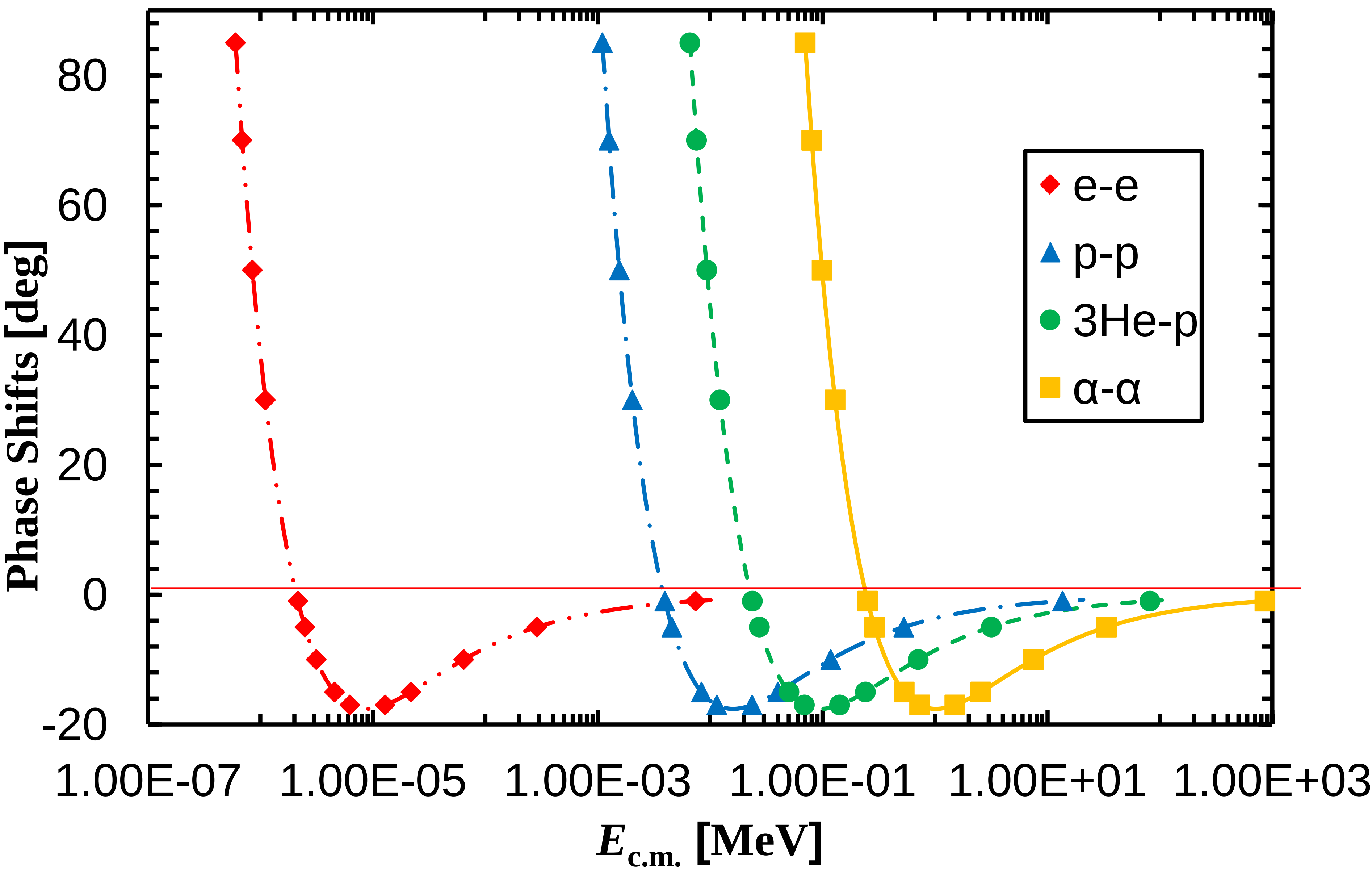
If we define the universal variables, then The Coulomb phase shifts of all the systems from e^--e^- to heavy ion systems are automatically obtained.

$\mathscr{R}(k) = kr$ Universal range

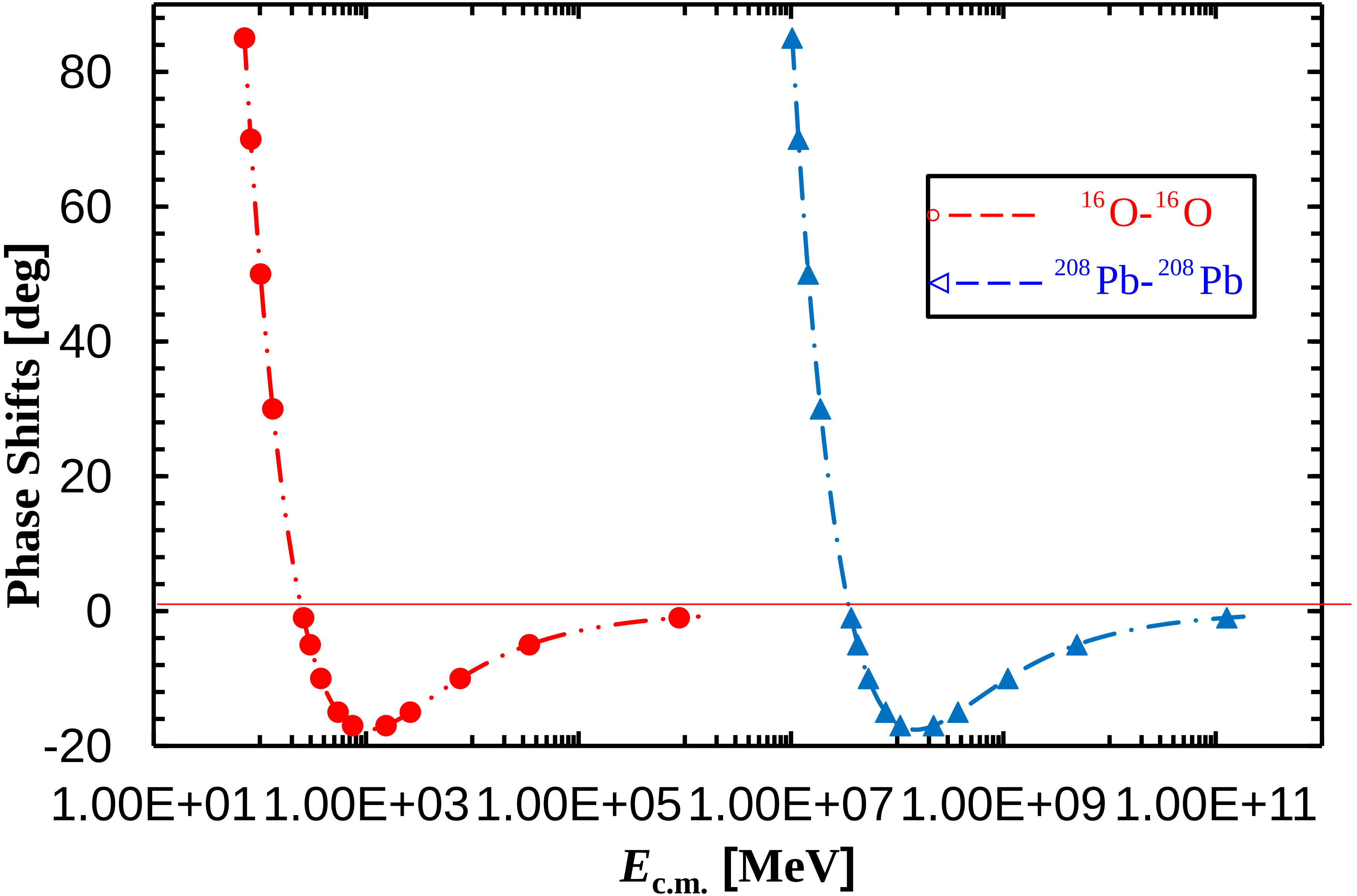
$\eta(k) = \frac{Z_1 Z_2 e^2 \nu_{12}}{k}$ Sommerfeld parameter

reduced mass $\nu_{12} = \frac{m_1 m_2}{m_1 + m_2}$

$\mathscr{F}(k) = 2kr_C$ universal asymptotic phase



Phase Shifts [deg]
80
60
40
20
0
-20
1.00E-07
1.00E-05
1.00E-03
1.00E-01
1.00E+01
1.00E+03
$E_{c.m.}$ [MeV]
e-e
p-p
3He-p
α-α



80
60
40
20
0
-20
Phase Shifts [deg]
1.00E+01
1.00E+03
1.00E+05
1.00E+07
1.00E+09
1.00E+11
E c.m. [MeV]
16O-16O
208Pb-208Pb

We conclude that the screened Coulomb potential with the unique range satisfies the *Lemma*, because, let us define the auxiliary potential

$$V^{\phi} = V^{C} - V^{R}$$

And for the potential

$$V = V^{S} + V^{C}$$

the total amplitude is given by the two-potential theory,

$$T = \varpi^{\phi}\varpi^{R\phi}t^{SR\phi}\omega^{R\phi}\omega^{\phi} + \varpi^{\phi}t^{R\phi}\omega^{\phi} + t^{\phi} \qquad (A)$$

with

$$t^{\phi}(k,k;E) = 0 \qquad Lemma \qquad (B)$$

$$t^{\phi}(p,p';E) = \langle p | \left(1 + t^{\phi}G_0\right)V^{\phi} | p' \rangle$$

$$\equiv \langle p | \varpi V^{\phi} | p' \rangle = \langle p | V^{\phi}\omega | p' \rangle \qquad (C)$$

Therefore the Møller operators ϖ, ω are the half-off-shell functions.

The Schroedinger equation for V^{R} satisfies the half-off-shell wave function and on shell phase shift. Therefore, the fully off-shell solution of Eq.(A): $T(p,p';E)$ is exactly obtained.

Conclusion

1) The generalization of the Faddeev equations offers
   a new tool for the neuclear reaction analysis.

2) Below the break up reaction, the E2Q is a unique method
   in the few - body problems.

3) From the E2Q, the long range interaction appears, where
   the Yukawa potential and the long range potential play
   complementary roles. Therefore, E2Q may open the
   pico size science.

4) The screening range of the Coulomb potential is a unique
   and a discrete band.
   We obtain the fully off - shell nuclear plus Coulomb amplitude.

Thank you very much for your attention.